

N O T I C E

THIS DOCUMENT HAS BEEN REPRODUCED FROM
MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT
CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED
IN THE INTEREST OF MAKING AVAILABLE AS MUCH
INFORMATION AS POSSIBLE

JPL PUBLICATION 81-107

(NASA-CR-168443) QUEUEING ANALYSIS OF
SYNCHRONOUS TIME DIVISION MULTIPLEXING WITH
INDIVIDUAL SOURCE BUFFERING (Jet Propulsion
Lab.) 68 p HC A04/MF A01 CSCL 17B

N82-17252

Unclas
G3/17 08912

Queueing Analysis of Synchronous Time Division Multiplexing with Individual Source Buffering

L. Clare



February 1, 1982

NASA

National Aeronautics and
Space Administration

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

JPL PUBLICATION 81-107

Queueing Analysis of Synchronous Time Division Multiplexing with Individual Source Buffering

L. Clare

February 1, 1982



National Aeronautics and
Space Administration

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

The research described in this publication was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

CONTENTS

ABSTRACT	1
I. INTRODUCTION	2
II. DEFINITION OF THE STDN MODEL	6
III. PACKET SYSTEM SIZE ANALYSIS	9
IV. DELAY ANALYSIS	22
Packet Delay	22
Message Delay	28
V. TRANSFORM EXPRESSIONS FOR THE CASE $N = \infty$	32
VI. NUMERICAL EXAMPLES	39
V. SUMMARY	41
REFERENCES	43
APPENDIX A: MESSAGE SYSTEM SIZE ANALYSIS	45
APPENDIX B: DISTRIBUTION OF ARRIVAL TIME MOD T	51

Figures

1. STDN Structure	54
2a. A Packet System Size Sample Function, Finite Packet Capacity	55
2b. A Message System Size Sample Function, Finite Message Capacity	55
3. The Semi-Markov Process Z_t	56
4. Packet System Size Density vs. Time	57
5. Mean System Size vs. Traffic Intensity for Various Normalized Slot Widths	58
6. Mean System Size vs. Traffic Intensity for Various System Size Capacities	59

CONTENTS (Contd.)

7.	Packet Blocking Probability vs. Traffic Intensity	60
8.	Mean Packet Delay vs. Traffic Intensity	61
9.	Mean Packet Delay vs. Throughput	62
10.	Actual Delay Densities	63

ABSTRACT

An analysis of Synchronous Time Division Multiplexing is presented. Packets of information arrive at the system as a compound Poisson process, and can be transmitted only during individual periodic intervals. Packet arrivals may be blocked (lost) if the system has a finite capacity and is congested. Using the theory of semi-regenerative processes, the distribution of the number of packets in the system (system size) is found. This nonstationary distribution is used to determine the complete system behavior, including the delay distributions, the blocking probability, and the density of the system size at arrival instants. Numerical examples illustrate applications of the results given.

I. INTRODUCTION

Spacecraft data is collected by means of a variety of sensors which typically operate simultaneously. The data from the various sensors must be funneled through a common telemetry channel. The current method of combining the data is by Synchronous Time Division Multiplexing (STDM).

This paper represents the initial phase of research in the study of Deep Space packet telemetry techniques [1]. This study, which is part of the NEEDS (NASA End-to-End Data System) program, will identify and compare various strategies for managing, packetizing and multiplexing spacecraft data. The baseline technique defined in [1] is STDM.

A unified approach to analyze the performance of STDM is given herein. Using this approach, several new results are obtained for a large class of arrival streams, particularly for the finite buffer capacity case.

The design engineer may use these results to completely predict system behavior, including channel utilization, probability of data loss due to buffer overflow, and queueing delay. By varying the design parameters of the model, the appropriate channel allocation and buffer sizing can be determined such that the performance will meet the prescribed criteria for each source. In addition, sensitivity analyses can be done by exploiting the generality of the arrival process model.

In STDM, each source is assigned a fixed sequence of time intervals during which it may transmit information. The assignment is predetermined and does not adapt to fluctuations in the traffic load. In the definition of STDM in this study, a structure is imposed on the time intervals assigned

to the sources, and is defined as follows. Time is divided into equal length frames. Each source is allotted a fixed contiguous portion of each frame, defined as a slot, which remains in a fixed ordered position from frame to frame. While the slot sizes are constant for a given source, they may vary between sources. The amount of information transmitted in a slot is called a packet (for that source).

It is clear that STDM provides excellent performance if the data arrival times are deterministic, one packet per frame. In the past, spacecraft data were collected in a predetermined fashion so that STDM was well suited for the environment. However, future missions will use source encoders as well as sensors that are triggered by random-time events. These will cause the information packets to arrive at random times, in which case STDM may not be as efficient as other alternatives. This paper investigates the performance of STDM with random time-of-arrival input streams.

In any multiplexing scheme the source data are colocated at the concentrator. This allows the possibility of sharing another finite resource: buffer space. However, in this paper we will assume each source has its own individual buffer. This important assumption implies that the queueing behavior of any source is independent of all other sources, and only one source needs to be considered. Since the STDM scheme considered here does not utilize any of the advantages due to the colocation of the sources, it is also referred to as Time Division Multiple Access (TDMA).

The analysis of the STDM queueing process presented here is based on the theory of semi-regenerative processes, assuming a Poisson message arrival stream. Messages will consist of groups of packets, where the distribution

of the group size is arbitrary (i.e. a compound Poisson packet arrival stream is assumed). The quantities of interest are: the number of packets or messages in the system ("system size"), the total time spent in the system ("delay"), and in the case of a finite capacity system, the probability of being blocked.

In this paper we give new results describing STDM/TDMA behavior, including the actual delay distributions and a complete characterization of intraframe system behavior for finite packet capacity systems. A list of previous work on STDM/TDMA is given by [2]-[8] and [13]-[17]. Birdsall et al. [13] and Dor [2] found the system size distribution at framing instants as well as the blocking probability for a finite capacity system. Chu [14] extended these results to the case of compound Poisson arrivals. Konheim [3], Hayes [4], Lam [5] and Rubin [6,7] all concentrated on the infinite buffer capacity case. Konheim [3] derived the packet system size distribution at framing instants and the mean virtual packet waiting time. Hayes [4] describes intra-frame behavior, and gives the actual waiting time distribution as well as the packet system size distribution. The first work to consider the packet transmission time within the frame was Lam's [5]; his paper gives the message system size distribution and mean message delay. Yan [8] extended Lam's work to allow a finite message capacity. Rubin [6,7] used a discrete-time model (with infinite buffer capacity) to obtain the system size and delay distributions. His model is more general than the one used here in that a source is allowed multiple contiguous slots in each frame. In addition, Rubin points out that the interarrival times may be correlated within the discrete time unit, so that a more general arrival process is allowed. Generalizations in other directions are given by [15]-[17].

The STDM model will be defined in the next section. Section III will present the fundamental analysis leading to the key result (28)-(29), the steady state packet system size density. This is used to find the packet system size at arrival instants (a non-Markov chain), as well as the blocking and truncation probabilities. Section IV contains the derivation of the packet and message delay distributions. Transform relations are given in Section V for the infinite capacity case. Numerical examples are then given in Section VI, followed by a concluding summary. The appendix contains an analysis of the message system size for a possibly finite message capacity system; this extends the works of Yan [8] and Lam [5].

II. DEFINITION OF THE STDM MODEL

In STDM, time is divided into equal length frames. Each frame is further divided into ordered slots, and one slot is assigned to each source. Let

T = frame duration

M = number of sources being multiplexed

δ_m = proportion of frame assigned to source m , $m = 1, 2, \dots, M$

so that

$\delta_m T$ = slot duration for the m th source, $m = 1, 2, \dots, M$

The STDM structure is illustrated in Figure 1. Frequently, STDM (or TDMA) is used with all the sources having the same slot size, so that $\delta_m = 1/M$.

As was previously mentioned, the queueing behavior for a given source is independent of all the other sources, so that only one needs to be considered. Define the time origin as though a slot for the source under consideration has just ended. Thus, the source may use the cross-hatched slots of duration $\delta_M T$ illustrated in Figure 1.

All subsequent definitions will refer only to the individual source under consideration. For example, the "packet system size" is the total number of packets in the system belonging to that source. In addition, the subscript on δ_M will be dropped, so that

δ = proportion of frame assigned to the source

Messages enter the system as a Poisson arrival process. Thus, letting

A_n = time of the n th message arrival

λ = message arrival rate, messages/time unit

we have that the interarrival times are independent and have the distribution

$$P(A_{n+1} - A_n \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0, \text{ any } n$$

Messages are assumed to consist of a positive integer number of packets.

Let

G_n = number of packets in the n th message

Then $\{G_n\}$ is a sequence of independent and identically distributed random variables with finite mean and an otherwise arbitrary density given by

$$g_\ell = P(G_n = \ell), \quad \ell = 1, 2, \dots, n = 1, 2, \dots$$

A packet is transmitted whenever the system is not empty at the beginning of a slot. If a packet arrives to an empty system during a slot interval, it must wait for the next slot before it can begin being transmitted (i.e. a complete slot is required for transmission).

The "system size" can be counted either in packets or in messages, and is defined to include both those in the buffer and that "in service" (i.e. being transmitted). Figure 2a illustrates a packet system size sample path, while Figure 2b shows the message system size for the same arrival stream. The sequences $\{R_n\}$ and $\{\tilde{R}_n\}$ represent the departure instants of packets and messages respectively. Each of the systems illustrated has a finite capacity. Packets or messages that are blocked are assumed to be

irretrievably lost. From Figure 2 it is clear that placing a finite constraint on the packet state space implies a different blocking procedure for the system than placing a finite constraint on the message state space.

The primary concern of this paper is the determination of the packet system size distribution (although the message system size distribution is derived in the appendix). Therefore we define for the body of this paper

X_t = packet system size at time t

N = capacity of the system in packets (possibly infinite)

For $N < \infty$ the implicit blocking procedure is as follows: if a message of G packets arrives to a system containing $N-L$ packets, and L is less than G , then L packets will be accepted and the remaining $G-L$ will be blocked (lost). Thus message integrity is not maintained, and several partial messages could be in the buffer at the same time. (The term "message" seems to imply an underlying mutual information between packets in the same arrival group, which may not be the case. The packets may actually contain independent information, with multipacket "messages" being used solely to model the burstiness of the arrival process.)

The next section begins by defining processes which are "embedded" within the $\{X_t\}$ process. These processes are then characterized, and provide the means for determining the distribution of X_t .

III. PACKET SYSTEM SIZE ANALYSIS

In this section we utilize the theory of semi-regenerative processes to find the distribution of the packet system size X_t . We will identify the processes and functions necessary for the development of the theory, and outline the approach leading to the main result ((28), (29)). Some immediate applications are then provided, including computation of the blocking probability.

The following processes are embedded within the process $\{X_t\}$: Let

R_n = time of the n th packet departure

$Y_n = X_{R_n^+}$ = packet system size just after the n th departure

and define

$$Z_t = Y_n \quad \text{for } R_{n-1} \leq t \leq R_n$$

Since the arrival process is Poisson, $\{Y_n\}$ is a (time-homogeneous) Markov Chain (MC), $\{(Y_n, R_n)\}$ is a Markov Renewal Process (MRP), and $\{Z_t\}$ is the minimal Semi-Markov Process (SMP) associated with $\{(Y_n, R_n)\}$. $\{X_t\}$ is a Semi-Regenerative Process (SRP) with respect to the MRP $\{(Y_n, R_n)\}$. These facts can be easily checked. For definitions of the above terms see [9].

An illustrative example showing the various processes defined above is given in Figure 3.

Before proceeding further, let us first define a convenient notation for convolutions of the density $\{g_k\}$ of the number of packets per message: let

$$g_k^{*j} = P(G_1 + G_2 + \dots + G_j = k) \quad , \quad k = 1, 2, \dots, j = 1, 2, \dots$$

and let

$$g_k^{*0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

Now consider the MC $\{Y_n\}$. Let the transition probabilities for $\{Y_n\}$ be denoted

$$y_{ij} = P(Y_{n+1} = j | Y_n = i) \quad 0 \leq i, j \leq N-1 \quad (1)$$

Then for $0 < i \leq N-1, 0 \leq j < N-1$

$$\begin{aligned} y_{ij} &= \sum_{k=0}^{j-i+1} P\left(k \text{ messages arrive in } [0, T] \text{ and } \sum_{n=1}^k G_n = j - i + 1\right) \\ &= \begin{cases} \sum_{k=0}^{j-i+1} e^{-\lambda T} \frac{(\lambda T)^k}{k!} g_{j-i+1}^{*k} & \text{if } j - i + 1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2) \end{aligned}$$

For $i = 0, 0 \leq j < N-1$ we have

$$\begin{aligned} y_{0j} &= E [P(Y_1 = j | Y_0 = 0, A_1, G_1)] \\ &= \sum_{r=1}^{j+1} g_r \int_0^\infty P(Y_1 = j | Y_0 = 0, A_1 = t, G_1 = r) \lambda e^{-\lambda t} dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{r=1}^{j+1} g_r \sum_{m=0}^{\infty} \left\{ \int_0^{T-\delta T} \sum_{k=0}^{j-r+1} e^{-\lambda(T-\tau)} \frac{[\lambda(T-\tau)]^k}{k!} g_{j-r+1}^{*k} \lambda e^{-\lambda(\tau+mT)} d\tau \right. \\
&\quad \left. + \int_{T-\delta T}^T \sum_{k=0}^{j-r+1} e^{-\lambda(2T-\tau)} \frac{[\lambda(2T-\tau)]^k}{k!} g_{j-r+1}^{*k} \lambda e^{-\lambda(\tau+mT)} d\tau \right\} \\
&= e^{-\lambda T} \sum_{r=1}^{j+1} g_r \sum_{k=0}^{j-r+1} \frac{g_{j-r+1}^{*k}}{(k+1)!} \left\{ (\lambda T)^{k+1} + \frac{1}{1 - e^{-\lambda T}} \left(e^{-\lambda T} [\lambda T(1+\delta)]^{k+1} \right. \right. \\
&\quad \left. \left. - (\lambda T \delta)^{k+1} \right) \right\} \tag{3}
\end{aligned}$$

The case $j = N - 1$ is found using

$$y_{iN-1} = 1 - \sum_{j=0}^{N-2} y_{ij} \tag{4}$$

or by summing (2) or (3) from $j = N - 1$ to infinity.

Define

$$\rho = \lambda \bar{G} T \tag{5}$$

to be the traffic intensity, where

$$\bar{G} = E(G_1)$$

is the mean number of packets in a message. By assumption $\bar{G} < \infty$.

The MC $\{Y_n\}$ is irreducible and aperiodic, and is positive recurrent if and only if either $N < \infty$ or $\rho < 1$. In this case the stationary distribution

$$\pi_j = \lim_{n \rightarrow \infty} P(Y_n = j | Y_0 = i) \quad j = 0, 1, \dots, N - 1$$

for the chain exists independent of i , is unique, and satisfies

$$\pi_j = \sum_{i=0}^{N-1} \pi_i y_{ij} \quad , \quad j = 0, 1, \dots, N-1 \quad (6)$$

$$\sum_{j=0}^{N-1} \pi_j = 1 \quad (7)$$

If $N < \infty$, Yan [8] gives an efficient method for evaluating $\{\pi_j\}$.

Now consider the MRP $\{(Y_n, R_n)\}$. Let

$$Q_{ij}(t) = P(Y_{n+1} = j, R_{n+1} - R_n \leq t | Y_n = i) \quad 0 \leq i, j \leq N-1 \quad (8)$$

be the Semi-Markov Kernel (SMK) for (Y, R) . Also, for i, j fixed let $\{R_n(i, j)\}$ be the sequence of times at which a departure occurs and j are in the system immediately thereafter, given that we start with i in the system. This is a delayed renewal process for each i, j . Let

$$N_{i,j}(t) = \sup_{n \geq 0} \{n: R_n(i, j) \leq t\} \quad (9)$$

be the associated counting process to the process $\{R_n(i, j)\}$. Let

$$m_{ij}(t) = E(N_{ij}(t)) \quad (10)$$

be the average number of visits to state j by time t , given we start in state i . Then $\{m_{ij}\}$ is the Markov Renewal Kernel (MRK) for the MRP (Y, R) . The MRK can be written in terms of the SMK (useful for transient results):

$$m_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^n(t) \quad (11)$$

where

$$Q_{ij}^n(t) = P(Y_{m+n} = j, R_{m+n} - R_m \leq t | Y_m = i)$$

is the n -stage transition distribution, which satisfies

$$Q_{ij}^{n+1}(t) = \sum_{k=0}^{N-1} \int_0^t Q_{kj}^n(t-s) dQ_{ik}(s) \quad (12)$$

For each i, j the process $\{R_n(i, j)\}$ is periodic, so that each state of (Y, R) is periodic with period T .

$R_1(j, j)$ is the recurrence time for state j . Note that we are speaking of the state space of (Y, R) , which is easiest to visualize in terms of $\{Z_t\}$. Thus $R_1(j, j)$ is the time Z_t spends between two successive visits to state j .

Let

$$\eta_j = \{E[R_1(j, j)]\}^{-1} \quad j = 0, 1, \dots, N-1 \quad (13)$$

be the mean recurrence rate for state j .

Let

$$m_j = E[R_1 | Y_0 = j] \quad j = 0, 1, \dots, N-1 \quad (14)$$

be the mean sojourn time in state j , i.e. the mean time Z_t spends in state j before going to any other state. Then ([9], p. 329)

$$\eta_j = \frac{\pi_j}{\bar{m}} \quad j = 0, 1, \dots, N-1 \quad (15)$$

where

$$\bar{m} = \sum_{i=0}^{N-1} \pi_i m_i \quad (16)$$

is the mean steady state sojourn time (for $\{Z_t\}$).

(It can be shown that the steady state probability density of $\{Z_t\}$ is given by $\{m_j \eta_j\}$, independent of t , even though the SMP $\{Z_t\}$ is periodic. In our examination of the SRP $\{X_t\}$, however, we will find that the periodicity of $\{X_t\}$ is evidenced by the periodicity of its steady state distribution.)

Having established the necessary machinery, we may finally investigate the SRP $\{X_t\}$. Denoting

$$P_i(\cdot) = P(\cdot | Y_0 = i) \quad ,$$

we have

$$P_i(X_t = j) = P_i(X_t = j, R_1 > t) + P_i(X_t = j, R_1 \leq t) \quad .$$

Now

$$P_i(X_t = j, R_1 \leq t) = \sum_{k=0}^{N-1} \int_0^t P_i(X_t = j | Y_1 = k, R_1 = s) dP_i(Y_1 = k, R_1 \leq s) \\ = \sum_{k=0}^{N-1} \int_0^t P_k(X_{t-s} = j) dQ_{ik}(s)$$

since $\{X_t\}$ is semiregenerative. Thus $\{X_t\}$ satisfies a Markov Renewal Equation (MRE):

$$P_i(X_t = j) = P_i(X_t = j, R_1 > t) + \sum_{k=0}^{N-1} \int_0^t P_k(X_{t-s} = j) dQ_{ik}(s) \quad (17)$$

The solution (which can be shown is unique) to this MRE is ([9], p. 324)

$$P_i(X_t = j) = \sum_{k=0}^{N-1} \int_0^t P_k(X_{t-s} = j, R_1 > t - s) dm_{ik}(s) \quad (18)$$

We are interested in the steady state behavior of $\{X_t\}$. If either $N < \infty$ or $\rho < 1$ then the (periodic) MRP (Y, R) is irreducible and recurrent, so we may apply the Key Renewal Theorem ([9], p. 334) to obtain

$$\lim_{m \rightarrow \infty} P_i(X_{\tau+mT} = j) = T \sum_{k=0}^{N-1} \eta_k \sum_{m=0}^{\infty} P_k(X_{\tau+mT} = j, R_1 > \tau + mT) \quad (19)$$

where $0 \leq \tau < T$ and $i, j = 0, 1, \dots, N-1$. Using (15) and defining

$$\lim_{m \rightarrow \infty} P_i(X_{\tau+mT} = j) = x_j(\tau) \quad , \quad 0 \leq \tau < T, j = 0, 1, \dots, N \quad (20)$$

we have: if $N < \infty$ or $\rho < 1$, then for $0 \leq \tau < T$

$$x_j(\tau) = \frac{T}{m} \sum_{k=0}^{N-1} \pi_k \sum_{m=0}^{\infty} p_k(X_{\tau+mT} = j, R_1 > \tau + mT) \quad (21)$$

$$j = 0, 1, \dots, N-1$$

and

$$x_N(\tau) = 1 - \sum_{j=0}^{N-1} x_j(\tau) \quad (22)$$

If $k > 0$, $j < N$ then

$$p_k(X_{\tau+mT} = j, R_1 > \tau + mT) = \begin{cases} \sum_{\ell=0}^{j-k} e^{-\lambda\tau} \frac{(\lambda\tau)^\ell}{\ell!} g_{j-k}^{*\ell} & \text{if } m = 0, j-k \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

while for $k = 0$, $j < N$

$$p_0(X_{\tau+mT} = j, R_1 > \tau + mT) = \sum_{\ell=0}^j g_j^{*\ell} \left\{ e^{-\lambda\tau} \frac{(\lambda\tau)^\ell}{\ell!} + \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} e^{-\lambda(\tau+\delta T)} \frac{[\lambda(\tau+\delta T)]^\ell}{\ell!} \right\} \quad (24)$$

For $j > 0$ the mean sojourn time is simply

$$m_j = T \quad (25)$$

while

$$m_0 = T \cdot P \text{ (at least 1 arrival in } [0, T - \delta T])$$

$$\begin{aligned}
 & + \sum_{m=1}^{\infty} (m+1)T \cdot P \left(\text{none arrive in } [0, mT - \delta T] \text{ and at least} \right. \\
 & \quad \left. 1 \text{ arrives in } [mT - \delta T, (m+1)T - \delta T] \right) \\
 & = T \left\{ 1 + \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \right\} \quad (26)
 \end{aligned}$$

Thus

$$\bar{m} = T \left\{ 1 + \pi_0 \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \right\} \quad (27)$$

Combining (21)-(24) we obtain the primary result: if $N < \infty$ or $\rho < 1$ then the steady state probability density of $\chi_{\tau+mT}$ exists, m integer, $0 \leq \tau < T$, and is given by

$$\begin{aligned}
 x_j(\tau) = \frac{T}{\bar{m}} \left\{ \pi_0 \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \sum_{\ell=0}^j g_j^{*\ell} e^{-\lambda(\tau+\delta T)} \frac{[\lambda(\tau + \delta T)]^\ell}{\ell!} \right. \\
 \left. + \sum_{k=0}^j \pi_k \sum_{\ell=0}^{j-k} g_{j-k}^{*\ell} e^{-\lambda\tau} \frac{(\lambda\tau)^\ell}{\ell!} \right\} \quad j = 0, 1, \dots, N-1 \quad (28)
 \end{aligned}$$

$$x_N(\tau) = 1 - \sum_{j=0}^{N-1} x_j(\tau) \quad (29)$$

where \bar{m} is given by (27).

The formulas (28), (29) provide the basis for the determination of the various quantities that describe STDN queueing behavior at steady state. Having found the "virtual" packet system size distribution, we next find the "virtual" truncation and blocking probabilities. We then complete the section by investigating these same quantities from the perspective of an "actual" arrival.

If a message arrives at a time $t = mT + \tau$ and finds $X = N - J$ packets in the system ($N = \text{capacity}$), then the message will be truncated if it contains more than J packets. Our definition of truncation includes the possibility of $J = 0$, i.e., the probability of being truncated includes the probability $x_N(\tau)$ that the entire message is lost. The steady state virtual message truncation density is given by

$$P_T(\tau) = \sum_{j=0}^N x_{N-j}(\tau) \sum_{\ell=j+1}^{\infty} g_{\ell} \quad 0 \leq \tau < T \quad (30)$$

where the term "virtual" is used to indicate that this event is conditioned on the arrival occurring at a time $\tau + mT$, m integer.

Consider the probability that a packet is blocked, where the packet is chosen uniformly from all packets which attempt to enter the system. Suppose the packet arrives within a message of length L packets, which arrives at time $t = \tau + mT$ and finds $N - J$ in the system. The packet will be blocked if $L > J$ and the packet is one of the $L - J$ packets which are not accepted. Since the packet is randomly chosen,

$$P(\text{packet arrives in a message of size } \ell) = \frac{\ell g_{\ell}}{\bar{G}} \quad (31)$$

and

$$P(\text{packet is one of the } (l-j) \text{ not accepted out of } l) = \frac{l-j}{l} \quad (32)$$

Therefore, the steady state virtual packet blocking probability is

$$P_B(\tau) = \sum_{j=0}^N x_{N-j}(\tau) \sum_{l=j+1}^{\infty} \frac{lg_l}{G} \frac{l-j}{l} \quad 0 \leq \tau < T \quad (33)$$

This represents an "average" virtual blocking probability, with no prior knowledge relating to a bias in the selection procedure. If an ordering exists on the packets within a message, and if the selection of which are blocked or not is based on the ordering, then the appropriate (nonuniform) distributions may be used to determine the blocking probabilities for each packet position.

In Appendix B it is shown that for large n , the arrival time within the frame $\tau \equiv A_n \bmod T$ tends to be uniformly distributed over $[0, T)$. Since the arrival process is memoryless, the unconditional density

$$\begin{aligned} \bar{x}_j &= \lim_{n \rightarrow \infty} P(X_{A_n} = j) \\ &= P(\text{arrival at steady state finds } j \text{ packets in system}) \\ j &= 0, 1, \dots, N \end{aligned}$$

can be obtained by averaging (28) and (29) over τ , τ uniform on $[0, T)$.

Thus the "actual" steady state packet system size density is

$$\bar{x}_j = \frac{1}{\lambda \bar{m}} \left\{ \pi_0 \frac{e^{-\lambda T}}{1 - e^{-\lambda T}} \sum_{\ell=0}^j g_j^{*\ell} \sum_{i=\ell+1}^{\infty} \frac{1}{i!} \left(e^{-\lambda T} [\lambda T(1 + \delta)]^i - [\lambda T \delta]^i \right) \right. \\ \left. + e^{-\lambda T} \sum_{k=0}^j \pi_k \sum_{\ell=0}^{j-k} g_{j-k}^{*\ell} \sum_{i=\ell+1}^{\infty} \frac{(\lambda T)^i}{i!} \right\} \quad j = 0, 1, \dots, N-1 \quad (34)$$

$$\bar{x}_N = 1 - \sum_{j=0}^{N-1} \bar{x}_j \quad (35)$$

where \bar{m} is given by (27).

By averaging (30) and (33) over τ we obtain the actual steady state message truncation and packet blocking probabilities respectively:

$$\bar{p}_T = \sum_{j=0}^N \bar{x}_{N-j} \sum_{\ell=j+1}^{\infty} g_{\ell} \quad (36)$$

$$\bar{p}_B = (\bar{G})^{-1} \sum_{j=0}^N \bar{x}_{N-j} \sum_{\ell=j+1}^{\infty} (\ell - j) g_{\ell} \quad (37)$$

The throughput s for the system is then

$$s = (1 - \bar{p}_B) \rho \quad (38)$$

The throughput s represents the average number of packet departures (or unblocked arrivals) per frame.

At equilibrium, the average unblocked packet arrival rate, $(1 - \bar{p}_B) \lambda \bar{G}$, equals the average packet departure rate, $1/\bar{m}$. Thus simpler expressions for the packet blocking probability and throughput are

$$\bar{p}_B = 1 - \frac{1}{\lambda \bar{G} \bar{m}} \quad (39)$$

$$s = \frac{T}{\bar{m}} \quad (40)$$

Before concluding this section, let us consider the "actual" packet system size when the messages consist of single packets. Evaluating (34) for this case we find: if $g_1 = 1$, then

$$\bar{x}_j = \frac{\pi_j}{\lambda \bar{m}}, \quad j = 0, 1, \dots, N-1$$

and

$$\bar{x}_N = \bar{p}_B = 1 - \frac{1}{\lambda \bar{m}}$$

This result could have been deduced from the following theorem [10]: for any stochastic system size process which changes only in unit steps, if either of

$$\alpha_j = \lim_{t \rightarrow \infty} P(\text{unblocked arrival at } t \text{ finds } j \text{ in system})$$

$$\pi_j = \lim_{t \rightarrow \infty} P(\text{departure at } t \text{ leaves } j \text{ in system})$$

exists, then so does the other and they are equal. Note that the message system size process satisfies the conditions of the theorem (recall Figure 2b) even in the case of multipacket message arrivals.

In the next section further utilization is made of equations (28) and (29). Both the packet and message steady state delay distributions are obtained by a straightforward application of these results.

IV. DELAY ANALYSIS

The packet delay is defined to be the total time the packet spends in the system, and similarly, the message delay is the time from its arrival until its last packet has completed transmission. This section presents results for the steady state packet and message delay distributions. Both the "virtual" and "actual" delay distributions are given. The term "virtual" signifies that the probabilities are conditioned on the arrival occurring at a time $t \equiv \tau \bmod T$, while the "actual" distribution is unconditional.

The results of this section require that an additional assumption be made on the STDM model defined in section II: the messages are transmitted in the same order that they are received in. That is, we assume a First-Come-First-Serve (FCFS) queueing discipline is used.

Packet Delay:

We will first find the steady state density of the virtual packet delay. This density will be representative of an "average" packet in exactly the same sense as discussed for the packet blocking probability. Thus, the order of service of packets in the same message can be assumed to be random (uniform).

Let $D(\tau)$ be the virtual packet delay for a packet which is contained in a message that arrives at a time $A \equiv \tau \bmod T$, $0 \leq \tau < T$. We set $D(\tau) = +\infty$ if the packet is blocked; the probability of this event is $P_B(\tau)$ (eqn. (33)).

The distribution of $D(\tau)$ is discrete with atoms at the points $D(\tau) = nT - \tau$, $n = 1, 2, \dots, N + 1$, and $+\infty$. We look for

$$P(D(\tau) = nT - \tau | D(\tau) < \infty) = \frac{P(D(\tau) = nT - \tau, D(\tau) < \infty)}{P(D(\tau) < \infty)} \quad (41)$$

where

$$P(D(\tau) < \infty) = 1 - P_B(\tau) \quad (42)$$

First consider the case $0 \leq \tau < T - \delta T$:

For $1 \leq n \leq N$

$$P(D(\tau) = nT - \tau | D(\tau) < \infty) = \frac{\sum_{j=0}^{n-1} P(X_\tau = j, L \geq n-j, (n-j)^{\text{th}} \text{ served})}{1 - P_B(\tau)} \quad (43)$$

where

L = size of the message that the randomly chosen packet arrived in

The density of L is given by (31). The probability that the packet will be served at a particular point in order $((n-j)^{\text{th}})$ within a group of L is $1/L$. Since the indicated events are independent,

$$P(D(\tau) = nT - \tau | D(\tau) < \infty) = [1 - P_B(\tau)]^{-1} \sum_{j=0}^{n-1} x_j(\tau) \sum_{\ell=n-j}^{\infty} \frac{\ell g_\ell}{\bar{G}} \frac{1}{\ell} \quad (44)$$

$$0 \leq \tau < T - \delta T, \quad 1 \leq n \leq N$$

Now consider the case $T - \delta T \leq \tau < T$. In this case the delay now depends on whether the packets already in the system all arrived since the beginning of the current slot. If at least one arrived before the slot began, then a departure will occur at the end of the slot; otherwise a departure will not occur until a frame later. Thus we have for $T - \delta T \leq \tau < T$

$$\begin{aligned}
 P(D(\tau) = nT - \tau | D(\tau) < \infty) \\
 = [1 - P_B(\tau)]^{-1} \left\{ \sum_{j=0}^{n-2} P(X_\tau = j, X_{T-\delta T} = 0, L \geq n-j-1, (n-j-1)^{\text{th}} \text{ served}) \right. \\
 \left. + \sum_{j=1}^{n-1} P(X_\tau = j, X_{T-\delta T} > 0, L \geq n-j, (n-j)^{\text{th}} \text{ served}) \right\} \quad (45)
 \end{aligned}$$

for $2 \leq n \leq N$, while $(n = N + 1)$

$$\begin{aligned}
 P(D(\tau) = (N + 1)T - \tau | D(\tau) < \infty) \\
 = [1 - P_B(\tau)]^{-1} \sum_{j=0}^{N-1} P(X_\tau = j, X_{T-\delta T} = 0, L \geq N-j, (N-j)^{\text{th}} \text{ served}) \quad (46)
 \end{aligned}$$

Since the arrival process is Poisson, it possesses the property of stationary independent increments, so that

$$\begin{aligned}
 P(X_\tau = j, X_{T-\delta T} = 0) = x_0(T - \delta T) P(j \text{ packet arrivals in } [0, \tau - (T - \delta T)]) \\
 j = 0, 1, \dots, N, \quad T - \delta T \leq \tau < T \quad (47)
 \end{aligned}$$

where from (28)

$$x_0(T - \delta T) = \frac{\pi_0 e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T} + \pi_0 e^{-\lambda T(1-\delta)}} \quad (48)$$

Separating independent events we may proceed as in the previous case, yielding:

For $T - \delta T \leq \tau < T$

$$\begin{aligned} P(D(\tau) = nT - \tau | D(\tau) < \infty) \\ = [1 - P_B(\tau)]^{-1} \left\{ \sum_{j=0}^{n-2} P(X_\tau = j, X_{T-\delta T} = 0) \sum_{\ell=n-j-1}^{\infty} \frac{g_\ell}{G} \right. \\ \left. + \sum_{j=1}^{n-1} [x_j(\tau) - P(X_\tau = j, X_{T-\delta T} = 0)] \sum_{\ell=n-j}^{\infty} \frac{g_\ell}{G} \right\} \quad (49) \end{aligned}$$

for $2 \leq n \leq N$, while

$$\begin{aligned} P(D(\tau) = (N+1)T - \tau | D(\tau) < \infty) \\ = [1 - P_B(\tau)]^{-1} \sum_{j=0}^{N-1} P(X_\tau = j, X_{T-\delta T} = 0) \sum_{\ell=N-j}^{\infty} \frac{g_\ell}{G} \quad (50) \end{aligned}$$

where

$$P(X_\tau = j, X_{T-\delta T} = 0) = x_0(T - \delta T) \sum_{i=0}^j e^{-\lambda[\tau - (T - \delta T)]} \frac{(\lambda[\tau - (T - \delta T)])^i}{i!} g_j^{*i} \quad (51)$$

and $x_0(T - \delta T)$ is given by (48).

Together, (44), (49) and (50) give the steady state virtual packet delay density.

Next we investigate the actual packet delay, which will be denoted simply D . As before we set $D = +\infty$ if the packet is blocked. We wish to find the distribution

$$P(D \leq t_0 | D < \infty)$$

for all t_0 . For each fixed t_0 , define (uniquely) n_0 and τ_0 so that

$$t_0 = n_0 T - \tau_0, \quad n_0 \text{ integer}, \quad 0 \leq \tau_0 < T \quad (52)$$

Let A be the time of arrival of the packet, and define τ so that $A \equiv \tau \pmod{T}$, $0 \leq \tau < T$. Conditioning on τ we find that

$$\text{for } \tau \geq \tau_0$$

$$\begin{aligned} P(D \leq t_0 | \tau, D < \infty) &= P(D(\tau) \leq t_0 | D(\tau) < \infty) \\ &= P(D(\tau) \leq n_0 T - \tau | D(\tau) < \infty) \end{aligned} \quad (53)$$

while for $\tau < \tau_0$

$$P(D \leq t_0 | \tau, D < \infty) = P(D(\tau) \leq (n_0 - 1)T - \tau | D(\tau) < \infty) \quad (54)$$

Since τ is uniform on $[0, T)$, unconditioning on τ gives

$$P(D \leq t_0 | D < \infty) = \frac{1}{T} \int_{\tau_0}^T P(D(\tau) \leq n_0 T - \tau | D(\tau) < \infty) d\tau \\ + \frac{1}{T} \int_0^{\tau_0} P(D(\tau) \leq (n_0 - 1)T - \tau | D(\tau) < \infty) d\tau \quad (55)$$

or changing variables

$$P(D \leq t_0 | D < \infty) = \frac{1}{T} \int_{(n_0-1)T}^{t_0} P(D(n_0 T - t) \leq t | D(n_0 T - \tau) < \infty) dt \\ + \frac{1}{T} \int_{t_0}^{n_0 T} P(D(n_0 T - t) \leq T - t | D(n_0 T - \tau) < \infty) dt \quad (56)$$

This is the desired distribution function, and can be shown to be continuous even at the points $t_0 = n_0 T$, n_0 integer. It is differentiable except when $t_0 = n_0 T$; using Leibniz' rule we obtain

$$\frac{d}{dt_0} P(D \leq t_0 | D < \infty) = \frac{1}{T} P(D(\tau_0) = t_0 | D(\tau_0) < \infty) \quad (57)$$

for $t_0/T \neq \text{integer}$,

where τ_0 is given by (52).

Summarizing, the actual packet delay, conditioned on the packet not being blocked, has a steady state distribution given by (56). This distribution is atomless, and has a density given simply by (57).

Message Delay:

The message delay analysis is similar to the packet delay analysis, and is actually simpler since the order in which the packets are served is not a concern. Therefore only the definitions and final results are presented here.

Let $\tilde{D}(\tau)$ be the virtual message delay for a message which arrives at time $A \equiv \tau \bmod T$, $0 \leq \tau < T$. Set $\tilde{D}(\tau) = +\infty$ if the message is completely blocked, thus,

$$P(\tilde{D}(\tau) = +\infty) = x_N(\tau) \quad (58)$$

The steady state conditional density of $\tilde{D}(\tau)$ is given by (59)-(63):

If $0 \leq \tau < T - \delta T$,

$$P(\tilde{D}(\tau) = nT - \tau | \tilde{D}(\tau) < \infty) = [1 - x_N(\tau)]^{-1} \sum_{j=0}^{n-1} x_j(\tau) g_{n-j} \quad (59)$$

for $1 \leq n \leq N - 1$, while

$$P(\tilde{D}(\tau) = NT - \tau | \tilde{D}(\tau) < \infty) = [1 - x_N(\tau)]^{-1} \sum_{j=0}^{N-1} x_j(\tau) \sum_{\ell=N-j}^{\infty} g_{\ell} \quad (60)$$

If $T - \delta T \leq \tau < T$,

$$\begin{aligned}
 & P(\tilde{D}(\tau) = nT - \tau | \tilde{D}(\tau) < \infty) \\
 &= [1 - x_N(\tau)]^{-1} \left\{ \sum_{j=0}^{n-2} P(X_\tau = 0, X_{T-\delta T} = 0) g_{n-j-1} \right. \\
 &\quad \left. + \sum_{j=1}^{n-1} [x_j(\tau) - P(X_\tau = 0, X_{T-\delta T} = 0)] g_{n-j} \right\} \quad (61)
 \end{aligned}$$

for $2 \leq n \leq N - 1$, while

$$\begin{aligned}
 & P(\tilde{D}(\tau) = NT - \tau | \tilde{D}(\tau) < \infty) \\
 &= [1 - x_N(\tau)]^{-1} \left\{ \sum_{j=0}^{N-2} P(X_\tau = 0, X_{T-\delta T} = 0) g_{N-j-1} \right. \\
 &\quad \left. + \sum_{j=1}^{N-1} [x_j(\tau) - P(X_\tau = 0, X_{T-\delta T} = 0)] \sum_{\ell=N-j}^{\infty} g_\ell \right\} \quad (62)
 \end{aligned}$$

and

$$\begin{aligned}
 & P(\tilde{D}(\tau) = (N + 1)T - \tau | \tilde{D}(\tau) < \infty) \\
 &= [1 - x_N(\tau)]^{-1} \sum_{j=0}^{N-1} P(X_\tau = j, X_{T-\delta T} = 0) \sum_{\ell=N-j}^{\infty} g_\ell \quad (63)
 \end{aligned}$$

where $P(X_\tau = j, X_{T-\delta T} = 0)$ is given by (51).

Let \tilde{D} be the actual message delay. The steady state distribution of \tilde{D} , conditioned on $\tilde{D} < \infty$, is atomless and has the density

$$\frac{d}{dt_0} P(\tilde{D} \leq t_0 | \tilde{D} < \infty) = \frac{1}{T} P(\tilde{D}(\tau_0) = t_0 | \tilde{D}(\tau_0) < \infty) \quad (64)$$

for $t_0/T \neq \text{integer}$, where τ_0 is given by (52).

In this section we determined the virtual packet and message delay distributions, expressed in terms of the virtual packet system size probabilities (28). The actual delay distributions were then given in terms of the virtual delay densities.

If we were only interested in the mean actual packet delay $E(D | D < \infty)$, it could be easily computed using Little's Result [11]:

$$sE(D | D < \infty) = T E(X) \quad (65)$$

where s is the throughput (38) and $E(X)$ is the mean of the actual packet system size density (34), (35). This simple result does not depend on any assumptions on the capacity N . Unfortunately, if $N < \infty$, any higher moments of the actual packet delay must be computed by a numerical integration involving the density (57). In addition, a finite capacity constraint imposes a similar numerical burden in obtaining any of the actual message delay moments. These numerical difficulties are alleviated in case $N = \infty$.

The next section concentrates on the infinite capacity system. Transform expressions are found for the steady state system size and delay distributions. These transforms are useful for determining moments; in particular, moments of the virtual delay distributions can be expressed in terms of those of the virtual packet system size distribution.

V. TRANSFORM EXPRESSIONS FOR THE CASE $N = \infty$

Further analysis is possible when the STDM system has an infinite buffer capacity. In this case simplifications occur in the transform representations of the various probability distributions of interest. The transforms can be used to obtain moments; some results are given in this section.

First consider the stationary distribution of the MC $\{Y_n\}$. Define for $|Z| \leq 1$

$$G^*(Z) = \sum_{\ell=1}^{\infty} g_{\ell} Z^{\ell} \quad (66)$$

to be the Z-transform of the number of packets per message. Also define for $|Z| \leq 1$

$$Y_e^*(Z) = \sum_{j=0}^{\infty} y_{0j} Z^j \quad (67)$$

and for $i > 0$

$$Y_b^*(Z) = \sum_{j=i-1}^{\infty} y_{ij} Z^{j-i+1} = \sum_{j=0}^{\infty} y_{1j} Z^j \quad (68)$$

The latter equality holds because y_{ij} is a function only of $j-i$ (see (2)) for $i > 0$. Using a straightforward generalization of Welch's result [12] to the case of compound Poisson arrivals, we find that the stationary distribution $\{\pi_j\}$ exists uniquely if $\rho < 1$. In this case the Z-transform of $\{\pi_j\}$ is given by ($|Z| \leq 1$)

$$\begin{aligned}
Y^*(Z) &= \sum_{j=0}^{\infty} \pi_j Z^j \\
&= \pi_0 \frac{Y_b^*(Z) - ZY_e^*(Z)}{Y_b^*(Z) - Z}
\end{aligned} \tag{69}$$

where

$$\pi_0 = \frac{(1 - \rho)(1 - e^{-\lambda T})}{\rho e^{-\lambda T(1-\delta)}} \tag{70}$$

Using (2) and (3) we find

$$Y_e^*(Z) = Z^{-1} e^{-\lambda T(1-G^*(Z))} \left\{ 1 - \frac{e^{-\lambda T(1-\delta)G^*(Z)} [1 - e^{-\lambda T(1-G^*(Z))}]}{1 - e^{-\lambda T}} \right\} \tag{71}$$

and

$$Y_b^*(Z) = e^{-\lambda T(1-G^*(Z))} \tag{72}$$

so that

$$Y^*(Z) = \frac{(1 - \rho) e^{-\lambda T\delta(1-G^*(Z))} [1 - e^{-\lambda T(1-G^*(Z))}]}{\rho [e^{-\lambda T(1-G^*(Z))} - Z]} \tag{73}$$

Now define the Z-transform of the steady state packet system size density at time $t = \tau + mT$ (large integer m) to be $(|Z| \leq 1, 0 \leq \tau < T)$

$$X^*(Z, \tau) = \sum_{j=0}^{\infty} x_j(\tau) Z^j \tag{74}$$

The mean steady state sojourn time is from (27) and (70)

$$\bar{m} = T/\rho \quad (75)$$

Using this and (28) we find for $\rho < 1$, $|Z| \leq 1$, $0 \leq \tau < T$

$$\begin{aligned} X^*(Z, \tau) &= (1 - \rho)e^{-\lambda(\tau+\delta T)(1-G^*(Z))} + \rho Y^*(Z)e^{-\lambda\tau(1-G^*(Z))} \\ &= \frac{(1 - \rho)(1 - Z)e^{-\lambda(\tau+\delta T)(1-G^*(Z))}}{e^{-\lambda T(1-G^*(Z))} - Z} \end{aligned} \quad (76)$$

Equation (76) provides a transform expression for the "virtual" packet system size probabilities at any time (during steady state conditions), and is primary to all subsequent developments in this section.

Note that $X^*(Z, T-\delta T)$ is the Pollaczek-Khinchin transform equation for the system size at departure instants of a bulk arrival M/D/1 queueing system. If we let $\delta = 0$, the Z-transforms of the packet system size just before and just after framing instants, $X^*(Z, T^-)$ and $X^*(Z, 0)$, agree with the results given by Konheim [3] and Hayes [4] respectively.

Differentiating (76) with respect to Z and evaluating at $Z = 1$ shows that the mean packet system size is the linear function of τ

$$E(X_{\tau+mT}) = (1 - \rho)\rho\delta + \rho\bar{Y} + \lambda\tau\bar{G} \quad (77)$$

where

$$\bar{Y} = \rho\delta - \frac{1}{2} + \frac{\bar{G}^2}{\bar{G}} \frac{1}{2(1 - \rho)} \quad (78)$$

where \bar{G} and \bar{G}^2 are the first and second moments of the group size G.

The Z-transform of the packet system size density as viewed by an actual arrival at steady state is defined as

$$\bar{X}^*(Z) = \sum_{j=0}^{\infty} \bar{x}_j Z^j \quad (79)$$

and is found using either (34) or by averaging (76) over τ to be ($\rho < 1$, $|Z| \leq 1$)

$$\begin{aligned} \bar{X}^*(Z) &= \frac{1 - e^{-\lambda T(1-G^*(Z))}}{\lambda T(1 - G^*(Z))} \left[(1 - \rho) e^{-\lambda T \delta(1-G^*(Z))} + \rho Y^*(Z) \right] \\ &= \frac{(1 - \rho) (1 - Z) e^{-\lambda T \delta(1-G^*(Z))} \left[1 - e^{-\lambda T(1-G^*(Z))} \right]}{\lambda T(1 - G^*(Z)) [e^{-\lambda T(1-G^*(Z))} - Z]} \end{aligned} \quad (80)$$

The mean (actual) packet system size is found to be

$$E(X) = \rho \delta + \frac{\overline{G^2}}{\overline{G}} \frac{\rho}{2(1 - \rho)} \quad (81)$$

We now determine the transforms of the delay distributions (assuming FCFS), using the results of section IV. Define for $0 \leq \tau < T$, $\text{Re}(s) \geq 0$

$$D^*(s, \tau) = E[e^{-sD(\tau)}] \quad (82)$$

to be the Laplace-Stieltjes Transform (LST) of the virtual packet delay steady state distribution. From (45)-(50) and (70) we find that

if $0 \leq \tau < T - \delta T$

$$D^*(s, \tau) = e^{-s(T-\tau)} \frac{1 - G^*(e^{-sT})}{(1 - e^{-sT}) \overline{G}} X^*(e^{-sT}, \tau) \quad (83)$$

while if $T - \delta T \leq \tau < T$

$$D^*(s, \tau) = e^{-s(T-\tau)} \frac{1 - G^*(e^{-sT})}{(1 - e^{-sT}) \bar{G}} \left\{ \chi^*(e^{-sT}, \tau) - (1 - \rho)(1 - e^{-sT}) e^{-\lambda[\tau - (T - \delta T)]} [1 - G^*(e^{-sT})] \right\} \quad (84)$$

Rubin [7] pointed out that

$$\hat{G}^*(Z) = \frac{1 - G^*(Z)}{(1 - Z) \bar{G}} \quad (85)$$

is the Z-transform of the Backward Renewal Time (BRT) \hat{G} of the renewal process with inter-renewal-times $\{G_n\}$. For an arbitrarily chosen packet, this BRT simply represents the number of packets in the same message that are served in front of the chosen packet.

Equations (83) - (85) allow moments of the virtual packet delay to be evaluated from moments of its components. For example, using

$$E(\hat{G}) = \frac{\overline{G^2} - \bar{G}}{2\bar{G}} \quad (86)$$

and (77), we find

$$E[D(\tau)] = \frac{T}{2} - \tau + \lambda\tau\bar{G} - \frac{\rho}{2} + T \frac{\overline{G^2}}{2(1 - \rho)\bar{G}} + \rho T \delta + \begin{cases} 0 & \text{if } \tau < T - \delta T \\ (1 - \rho)T & \text{if } \tau \geq T - \delta T \end{cases} \quad (87)$$

Now consider the LST of the actual packet delay, defined as

$$D^*(s) = E[e^{-sD}] \quad (88)$$

This can be determined from (83) and (84) by unconditioning on τ :

$$D^*(s) = \frac{(1 - \rho)(1 - e^{-sT}) \hat{G}^*(e^{-sT}) e^{-s\delta T}}{sT - \lambda T [1 - G^*(e^{-sT})]} \quad (89)$$

The mean actual packet delay is

$$E(D) = \delta T + T \frac{\overline{G^2}}{2(1 - \rho) \overline{G}} \quad (90)$$

Unfortunately, a simple relationship between $D^*(s)$ and $\bar{X}^*(s)$ does not occur, except in the case of single packet messages. If $g_1 = 1$, we obtain the usual M/G/1 result

$$D^*(\lambda - \lambda Z) = \bar{X}^*(Z) \quad (91)$$

("generalized Little's Result").

Next define the LST of the virtual message delay as

$$\tilde{D}^*(s, \tau) = E[e^{-s\tilde{D}(\tau)}] \quad , \quad 0 \leq \tau < T \quad (92)$$

From (59)-(63) we find

$$\tilde{D}^*(s, \tau) = e^{s\tau} G^*(e^{-sT}) \chi^*(e^{-sT}, \tau) \cdot \begin{cases} 1 & \text{if } \tau < T - \delta T \\ e^{-sT + \lambda T [1 - G^*(e^{-sT})]} & \text{if } \tau \geq T - \delta T \end{cases} \quad (93)$$

Define the LST of the actual message delay to be

$$\tilde{D}^*(s) = E[e^{-s\tilde{D}}] \quad (94)$$

Averaging (93) over τ we find

$$\tilde{D}^*(s) = \frac{(1 - \rho)(1 - e^{-sT}) G^*(e^{-sT}) e^{sT(1-\delta)}}{sT - \lambda T [1 - G^*(e^{-sT})]} \quad (95)$$

The mean actual message delay is

$$E(\tilde{D}) = \delta T + T\bar{G} - \frac{T}{2} + \frac{T\rho\bar{G}^2}{2(1 - \rho)\bar{G}} \quad (96)$$

This concludes the analysis of the STDM/TDMA system. In the next section the various results presented will be illustrated by numerical examples.

VI. NUMERICAL EXAMPLES

In this section several examples are presented in order to illustrate applications of the preceding theory. All cases refer to packet behavior at steady state. Examples of message behavior are contained in Yan [8] and Lam [5].

Figure 4 gives an example of the steady state packet system size probabilities versus time for each possible state, computed from (28) and (29). A vertical slice taken at a particular time $t = mT + \tau$ will yield the complete density $\{x_j(\tau)\}$ for that time. Also included on the graph is the virtual packet blocking probability $P_B(\tau)$, represented as a dashed line. The nonstationary (periodic) nature of the system size process is clearly evident. The example used for Figure 4 is a source allotted $\delta = .5$ of each frame, with traffic intensity $\rho = .85$ and a capacity constraint of $N = 5$ packets. The message length is fixed at 2 packets, so that $g_2 = 1$. Note that this causes $\bar{x}_1 < \bar{x}_2 < \bar{x}_0$.

Figure 5 shows the mean system size versus traffic intensity for the simple case of single packet messages and an infinite capacity. The curves are parameterized by the slot-to-frame ratio δ , which determines the degree of "funneling" in the system. This figure indicates that as δ is decreased, its incremental impact on the system behavior decreases rapidly.

Figures 6, 7 and 8 were generated for a source allotted $\delta = .01$, with either $g_8 = 1$ or $g_2 = 1$ (i.e. packets arrive in either groups of 8 or in groups of 2 respectively). For each of these configurations the system capacity is either 10, 20, 40, or an infinite number of packets.

Figure 6, like Figure 5, illustrates the mean ("actual") system size as a function of the traffic intensity ρ . We see that more congestion occurs for the more bursty input stream ($g_8 = 1$). The two cases cross over (same N) for large ρ values due to the relatively larger number of packets blocked in the case $g_8 = 1$ (see next figure).

The packet blocking probabilities \bar{P}_B are given in Figure 7 as a function of ρ . Notice that the $g_8 = 1, N = 40$ and $g_2 = 1, N = 10$ cases coincide, confirming the intuitive notion that a system with traffic four times as bursty requires four times the capacity to obtain the same \bar{P}_B .

Figure 8 shows the mean packet delays versus ρ . These curves were generated by dividing the mean system size (Figure 5) by the unblocked arrival rate (Little's Result); i.e., the mean is conditioned on $D < \infty$.

In the next example we consider two input streams with the same mean number of packets/message \bar{G} , but which differ in higher moments. Specifically, in one case $g_4 = 1$, while in the other we have $g_2 = 1/2$ and $g_4 = g_8 = 1/4$. The mean (unblocked) packet delay versus the throughput s is given by Figure 9. A smaller delay is attained by the zero variance case, $g_4 = 1$. Note that no crossover occurs for fixed N as it did in Figures 6 and 8, because s is used instead of ρ for the abscissa.

Figure 10 presents the actual packet and message delay densities for the same example as was used for Figure 4. Because of (57), this graph can also be used to determine the virtual delay densities.

V. SUMMARY

An exact analysis of the STDM/TDMA system with Poisson message arrivals has been given. The results were obtained by rigorously defining the underlying processes involved and then applying the theory of semi-regenerative processes. Steady state packet and message queueing behavior is predicted for possibly capacity-limited systems.

The embedded chain is defined as the system size at departure instants. The transition probability matrix is given, from which the stationary distribution can be obtained using either Yan's [8] method ($N < \infty$) or transform methods ($N = \infty$). The steady state system size at all times is then determined, and is shown to be time-periodic. By averaging over the period, we obtain the system size density as viewed by a typical message or packet arrival.

The system size densities are used to obtain the blocking and truncation probabilities (if appropriate). In addition, the delay distributions are expressed in terms of the packet system size density.

Numerical examples illustrate the probabilistic behavior of the STDM/TDMA system. The periodic nature of the queueing process is exemplified, and its effect on the delay density. Other examples present the mean system sizes, mean delays and blocking probabilities for various system parameter values.

The results of this paper will enable a performance prediction of the STDM/TDMA system. Considerable flexibility is allowed in modelling the packet arrival process. The design engineer can determine what buffer

size and allowable traffic load is required to maintain given blocking and delay constraints.

Acknowledgement

The author wishes to express his gratitude to Izhak Rubin and Tsun-Yee Yan for the many helpful discussions in connection with this work.

REFERENCES

1. Clare, L. P., "Initial Progress Report: Packet Telemetry Executive Control Study," JPL IOM 331-81-024, March 29, 1981 (an internal document).
2. Dor, N. M., "Guide to the Length of Buffer Storage Required for Random (Poisson) Input and Constant Output Rates," IEEE Trans. Electronic Computers (Short Notes), vol. EC-16, pp. 683-684, October 1967.
3. Konheim, A. G., "Service Epochs in a Loop System," MRI Symp. Proc., Vol. 22, Computer-Communications Networks and Teletraffic, Poly. Inst. Brooklyn, April 1972, pp. 125-143.
4. Hayes, J.F., "Performance Models of an Experimental Computer Communication Network," Bell Systems Tech. Journal #53, February 1974, pp. 225-259.
5. Lam, S. S., "Delay Analysis of a Time Division Multiple Access (TDMA) Channel," IEEE Trans. on Communications, COM-25, pp. 1489-1494, December 1977.
6. Rubin, I., "Message Delays in FDMA and TDMA Communication Channels," IEEE Trans. on Communications, COM-27, pp. 769-777, May 1979.
7. Rubin, I., "Access-Control Disciplines for Multi-Access Communication Channels: Reservation and TDMA Schemes," IEEE Trans. on Information Theory, IT-25, pp. 516-536, September 1979.
8. Yan, T. Y., "Queueing Analysis of Time Division Multiple Access Channel with Finite Buffer Capacity," JPL IOM 331-81-044A, February 12, 1981 (an internal document).

9. Cinlar, E., Introduction to Stochastic Processes, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.
10. Kleinrock, L., Queueing Systems Vol. I: Theory, Wiley-Interscience, New York, 1975, p. 176.
11. Little, J. D. C., "A Proof of the Queueing Formula $L = \lambda w$," Operations Res., Vol. 9, pp. 383-387, 1961.
12. Welch, P. D., "On a Generalized M/G/1 Queueing Process in which the First Customer of Each Busy Period Receives Exceptional Service," Operations Res., Vol. 12, 1964, pp. 736-752.
13. Birdsall, Ristenbatt, and Weinstein, "Analysis of Asynchronous Time Multiplexing of Speech Sources," IRE Trans. on Comm. Sys., Vol. CS-10, pp. 390-397, Dec. 1962.
14. Chu, W. W., "Buffer Behavior for Batch Poisson Arrivals and Single Constant Output," IEEE Trans. Comm. Tech., Vol. COM-18, pp. 613-618, Oct. 1970.
15. Chu, W. W., "Buffer Behavior for Poisson Arrivals and Multiple Synchronous Constant Outputs," IEEE Trans. Computers, Vol. C-19, pp. 530-534, June 1970.
16. Kekre and Saxena, "Analysis of Buffer Behaviour for Multiplexing of Data with Burst-Poisson Arrivals in Analog Speech Signals," J. Instn. Elect. and Telecom. Engrs., Vol. 25, pp. 407-414, 1979.
17. Shathikumar, J. G., "On the Buffer Behavior with Poisson Arrivals, Priority Service, and Random Server Interruptions," IEEE Trans. on Computers, Vol. C-30, Oct. 1981.
18. Revuz, D., Markov Chains, North Holland Publishing Co., Amsterdam, 1975.

APPENDIX A: MESSAGE SYSTEM SIZE ANALYSIS

The STDM/TDMA model is essentially the same as for the packet system size analysis. The parameters T , δ , λ , $\{g_\ell\}$ and ρ are defined as before. The remaining parameters are defined similarly except that they are measured in messages rather than packets. For example,

N = capacity of the system in messages

X_t = system size in messages at time t

R_n = time of n th message departure

etc. It is hoped that this duplication of notation does not cause confusion.

Note that for $N < \infty$ the system inherently operates in a different manner than in the packet capacity-limited case. Messages that arrive to a full system are completely blocked, so that messages which are accepted into the system are transmitted in their entirety. If $N = \infty$ the system operation is identical to that of the previous model, except now messages are counted instead of packets.

The discrete process $\{Y_n\}$, $Y_n = X_{R_n}^+$ is a MC with transition probabilities (see Yan [8]):

$$y_{ij} = \begin{cases} \sum_{\ell=1}^{\infty} g_\ell e^{-\lambda \ell T} \frac{(\lambda \ell T)^{j-i+1}}{(j-i+1)!} & j - i + 1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (A.1)$$

$j < N - 1$:

$$y_{0j} = \sum_{\ell=1}^{\infty} g_{\ell} \frac{e^{-\lambda \ell T}}{(j+1)!} \left\{ (\lambda \ell T)^{j+1} + \frac{1}{1 - e^{-\lambda T}} \left(e^{-\lambda T} [\lambda T(\ell + \delta)]^{j+1} - [\lambda T(\ell - 1 + \delta)]^{j+1} \right) \right\} \quad (\text{A.2})$$

and for any i

$$y_{iN-1} = 1 - \sum_{j=0}^{N-2} y_{ij} \quad (\text{A.3})$$

The MC $\{Y_n\}$ will possess a stationary distribution $\{\pi_n\}$ if either $N < \infty$ or $\rho < 1$, and will satisfy equations (6) and (7).

The mean sojourn times are

$$m_0 = T \left(\bar{G} + \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \right) \quad (\text{A.4})$$

and for $j > 0$

$$m_j = \bar{G} T \quad (\text{A.5})$$

so that

$$\bar{m} = T \left(\bar{G} + \pi_0 \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \right) \quad (\text{A.6})$$

Since X_t is semi-regenerative, (21) and (22) will remain true. We now have for $0 \leq \tau < T$:

$$P_0(X_\tau = j, R_1 > \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^j}{j!} \quad (\text{A.7})$$

while for $m > 0$ (m integer)

$$P_0(X_\tau = j, R_1 > \tau + mT) = P(G_1 > m) e^{-\lambda(\tau+mT)} \frac{[\lambda(\tau + mT)]^j}{j!} + \sum_{\ell=1}^{\infty} g_\ell e^{-\lambda[mT+\tau]} \frac{(\lambda[(\ell-1)T + \delta T + \tau])^j}{j!} \quad (\text{A.8})$$

and for $k > 0, m \geq 0$

$$P_k(X_{\tau+mT} = j, R_1 > \tau + mT) = \begin{cases} e^{-\lambda(\tau+mT)} \frac{[\lambda(\tau + mT)]^{j-k}}{(j-k)!} P(G_1 > m) & j - k \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.9})$$

Combining (21), (22), (A.6)-(A.9) we find that the steady state message size density at time $t = \tau + mT, 0 \leq \tau < T$ is

$$x_j(\tau) = \frac{T}{m} \left\{ \pi_0 \frac{e^{-\lambda T(1-\delta)}}{1 - e^{-\lambda T}} \sum_{\ell=1}^{\infty} g_\ell e^{-\lambda[(\ell-1)T + \delta T + \tau]} \frac{(\lambda[(\ell-1)T + \delta T + \tau])^j}{j!} + \sum_{k=0}^j \pi_k \sum_{m=0}^{\infty} P(G_1 > m) e^{-\lambda(mT+\tau)} \frac{[\lambda(mT + \tau)]^{j-k}}{(j-k)!} \right\} \quad (\text{A.10})$$

$0 \leq j < N$, and

$$x_N(\tau) = 1 - \sum_{j=0}^{N-1} x_j(\tau) \quad (\text{A.11})$$

Of course, there is no message truncation, and $x_N(\tau)$ represents the virtual message blocking probability.

Averaging (A.10) over τ (uniform) we find after simplification

$$\bar{x}_j = \frac{1}{\lambda \bar{m}} \sum_{k=0}^j \pi_k \sum_{i=j}^{N-1} y_{ki} \quad 0 \leq j < N \quad (\text{A.12})$$

But it can be easily shown that

$$\sum_{k=0}^j \pi_k \sum_{i=j}^{N-1} y_{ki} = \pi_j \quad (\text{A.13})$$

so that the steady state message system size density as viewed by arrivals is simply

$$\bar{x}_j = \frac{\pi_j}{\lambda \bar{m}} \quad 0 \leq j < N \quad (\text{A.14})$$

with the actual message blocking probability

$$\begin{aligned} \bar{x}_N &= 1 - \sum_{j=0}^{N-1} \bar{x}_j \\ &= 1 - \frac{1}{\lambda \bar{m}} \end{aligned} \quad (\text{A.15})$$

The system throughput is

$$s = (1 - \bar{x}_N) \rho = \frac{\rho}{\lambda \bar{m}} \quad (\text{A.16})$$

The Case $N = \infty$:

If the system has an infinite buffer capacity, Lam [6] showed that ($|Z| \leq 1$)

$$Y^*(Z) = \pi_0 \frac{S_b^*(\lambda - \lambda Z) - Z S_e^*(\lambda - \lambda Z)}{S_b^*(\lambda - \lambda Z) - Z} \quad (\text{A.17})$$

where

$$\pi_0 = \frac{(1 - \rho) (1 - e^{-\lambda T})}{\lambda T e^{-\lambda T(1-\delta)}} \quad (\text{A.18})$$

and where

$$S_b^*(u) = G^*(e^{-uT}) \quad (\text{A.19})$$

$$S_e^*(u) = \frac{\lambda G^*(e^{-uT})}{\lambda - u} \left[1 - \frac{e^{-(\lambda - u)(T - \delta T)} (1 - e^{-uT})}{1 - e^{-\lambda T}} \right] \quad (\text{A.20})$$

are the Laplace-Stieltjes transforms of the message "service times", conditioned on whether the message arrives to a busy or an empty system.

Combining (A.17)-(A.20) yields

$$Y^*(Z) = \frac{(1 - \rho) G^*(e^{-\lambda T(1-Z)})}{\lambda T e^{-\lambda T(1-\delta)(1-Z)}} \frac{1 - e^{-\lambda T(1-Z)}}{G^*(e^{-\lambda T(1-Z)}) - Z} \quad (\text{A.21})$$

Using (A.10) we find ($|Z| \leq 1$)

$$\begin{aligned} X^*(Z, \tau) &= (1 - \rho) e^{-\lambda(\tau - T + \delta T)(1-Z)} G^*(e^{-\lambda T(1-Z)}) \\ &\quad + \lambda T e^{-\lambda T(1-Z)} Y^*(Z) \frac{1 - G^*(e^{-\lambda T(1-Z)})}{1 - e^{-\lambda T(1-Z)}} \end{aligned} \quad (\text{A.22})$$

The mean steady state message system size at time $t = \tau + mT$, m integer, $0 \leq \tau < T$ is

$$E(X_{\tau+mT}) = \lambda(\tau - T + \delta T) + \rho[\bar{Y} + 1 - \rho + \lambda T(1 - \delta)] + \frac{(\lambda T)^2 (\bar{G}^2 - \bar{G})}{2} \quad (\text{A.23})$$

where

$$\bar{Y} = \rho - \frac{\lambda T}{2} + \lambda T \delta + \frac{(\lambda T)^2 \bar{G}^2}{2(1 - \rho)} \quad (\text{A.24})$$

By summing (A.14) over all j we find that $\lambda \bar{m} = 1$ and that

$$\bar{x}_j = \pi_j \quad \text{for all } j,$$

or

$$\bar{X}^*(Z) = Y^*(Z) \quad (\text{A.25})$$

APPENDIX B: DISTRIBUTION OF ARRIVAL TIME MOD T

It is intuitively clear that the limiting distribution of the arrival time within a frame is uniform when the arrival process is Poisson. A formal proof of this fact is given here.

As before we let

$$A_n = \text{time of the } n^{\text{th}} \text{ arrival}$$

and assume that arrivals form a Poisson process of rate λ . Let

$$I_n = A_n - A_{n-1}$$

be the interarrival times, and let

$$\tau_n = A_n - T[A_n/T]$$

be the time from the last frame instant to A_n , where T is the frame duration and notationally

$$[x] = \text{greatest integer less than or equal to } x$$

It is clear that the sequence of random variables $\{\tau_n\}$, defined on the state space $[0, T)$, forms a Markov Chain (MC). We compute the transition probabilities

$$P(\tau_{n+1} \leq t_1 \mid \tau_n = t_0)$$

by considering two separate cases.

$$i) \quad t_1 > t_0$$

$$P(\tau_{n+1} \leq t_1 \mid \tau_n = t_0) = P(0 \leq I_{n+1} \leq t_1 - t_0)$$

$$+ \sum_{m=1}^{\infty} P(mT - t_0 \leq I_{n+1} \leq mT - t_0 + t_1)$$

$$= 1 - e^{-\lambda(t_1 - t_0)}$$

$$+ \frac{e^{-\lambda(t_0 - t_1)} [1 - e^{-\lambda t_1}]}{1 - e^{-\lambda T}}$$

ii) $t_1 \leq t_0$

$$P(\tau_{n+1} \leq t_1 \mid \tau_n = t_0) = \sum_{m=1}^{\infty} P(mT - t_0 \leq I_{n+1} \leq mT - t_0 + t_1)$$

$$= \frac{e^{-\lambda(T - t_0)} [1 - e^{-\lambda t_1}]}{1 - e^{-\lambda T}}$$

Given the distribution of τ_n , the distribution of τ_{n+1} can be found via

$$P(\tau_{n+1} \leq t_1) = \int P(\tau_{n+1} \leq t_1 \mid \tau_n = t_0) dP(\tau_n \leq t_0)$$

Suppose τ_n is uniformly distributed on $[0, T)$. Then for $t_1 \in [0, T)$

$$P(\tau_{n+1} \leq t_1) = \int_0^{t_1} [1 - e^{-\lambda(t_1 - t_0)}] \frac{dt_0}{T} \\ + \int_{t_1}^T \frac{e^{-\lambda(T - t_0)} [1 - e^{-\lambda t_1}]}{1 - e^{-\lambda T}} \frac{dt_0}{T}$$

$$= \frac{t_1}{T}$$

so that τ_{n+1} is also uniform on $[0, T)$. Thus the uniform distribution is stationary for the MC. By Proposition 3.2.10 of [18] we know the stationary distribution is unique, and by Orey's Theorem (Theorem 6.2.8 of [18]) we know the distribution of the MC converges to it.

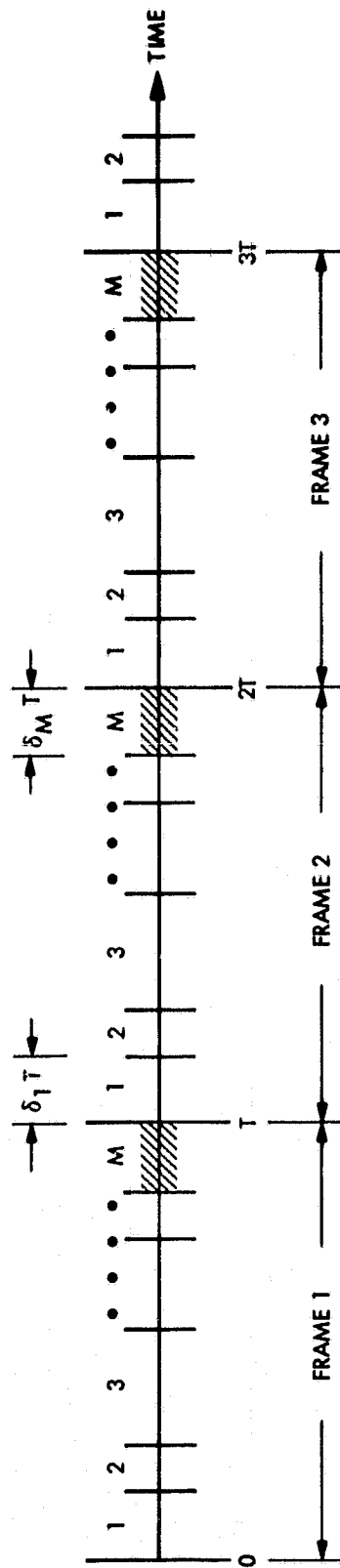


Figure 1. STDM Structure

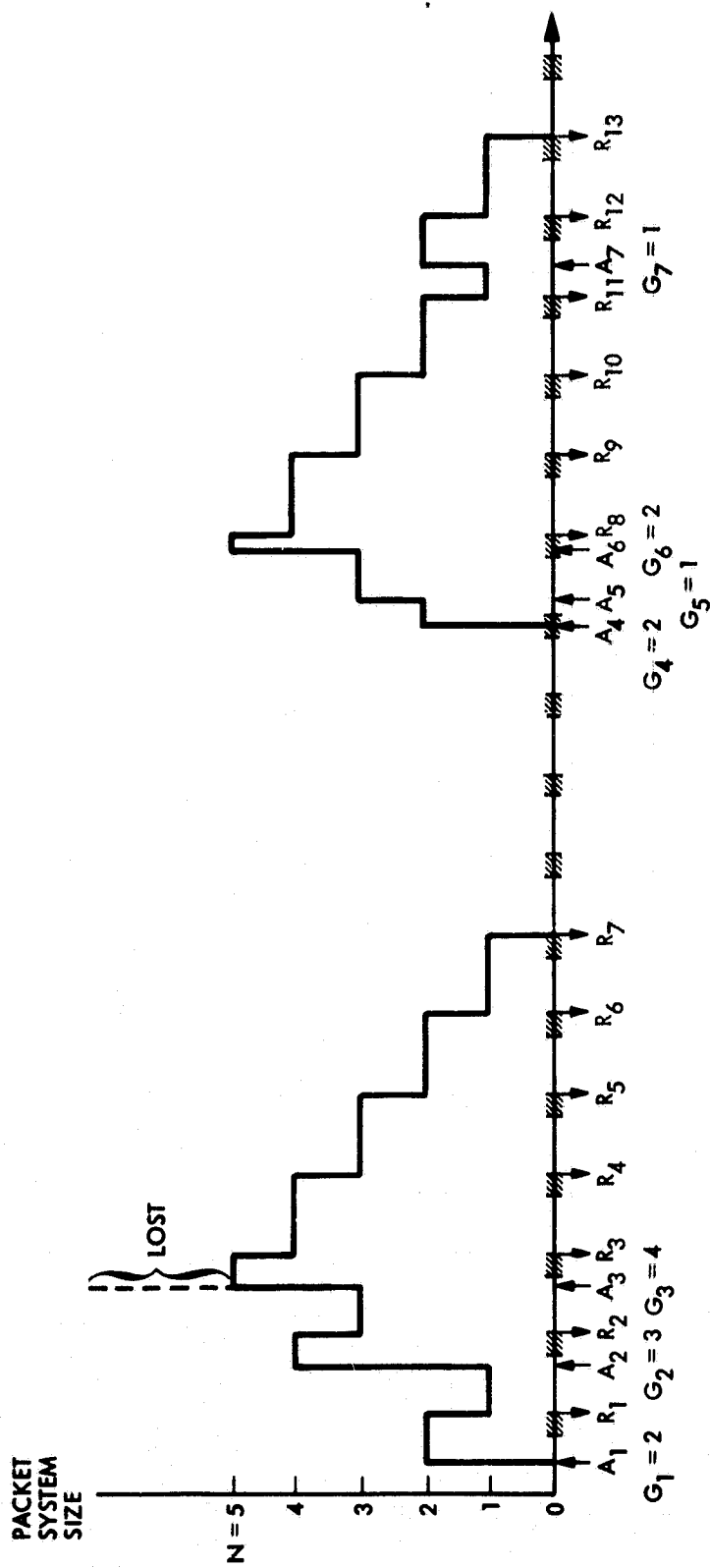


Figure 2a. A Packet System Size Sample Function, Finite Packet Capacity

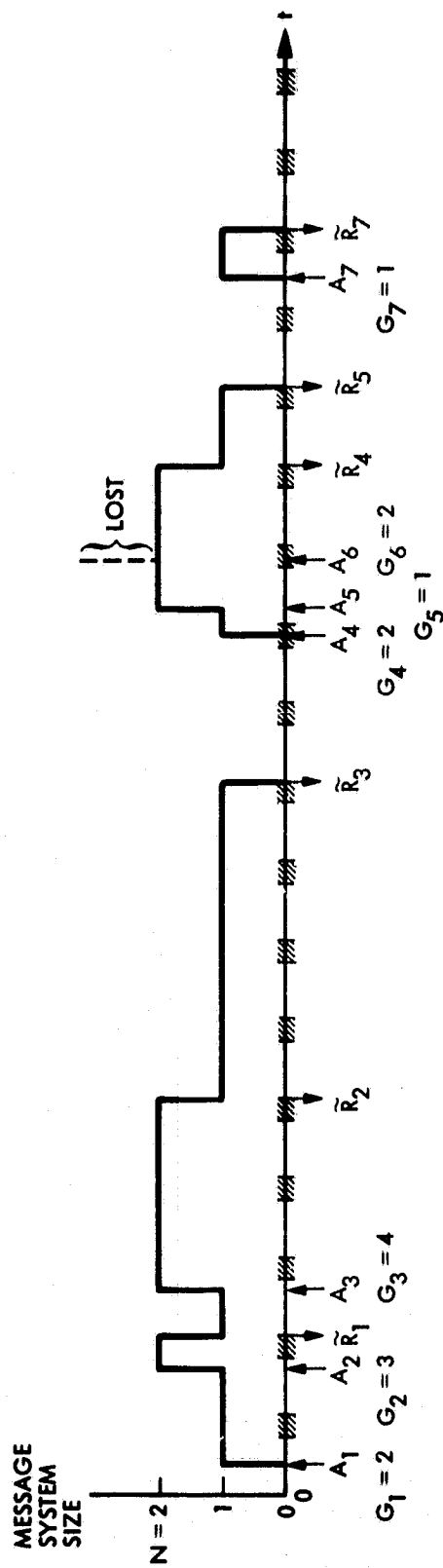


Figure 2b. A Message System Size Sample Function, Finite Message Capacity

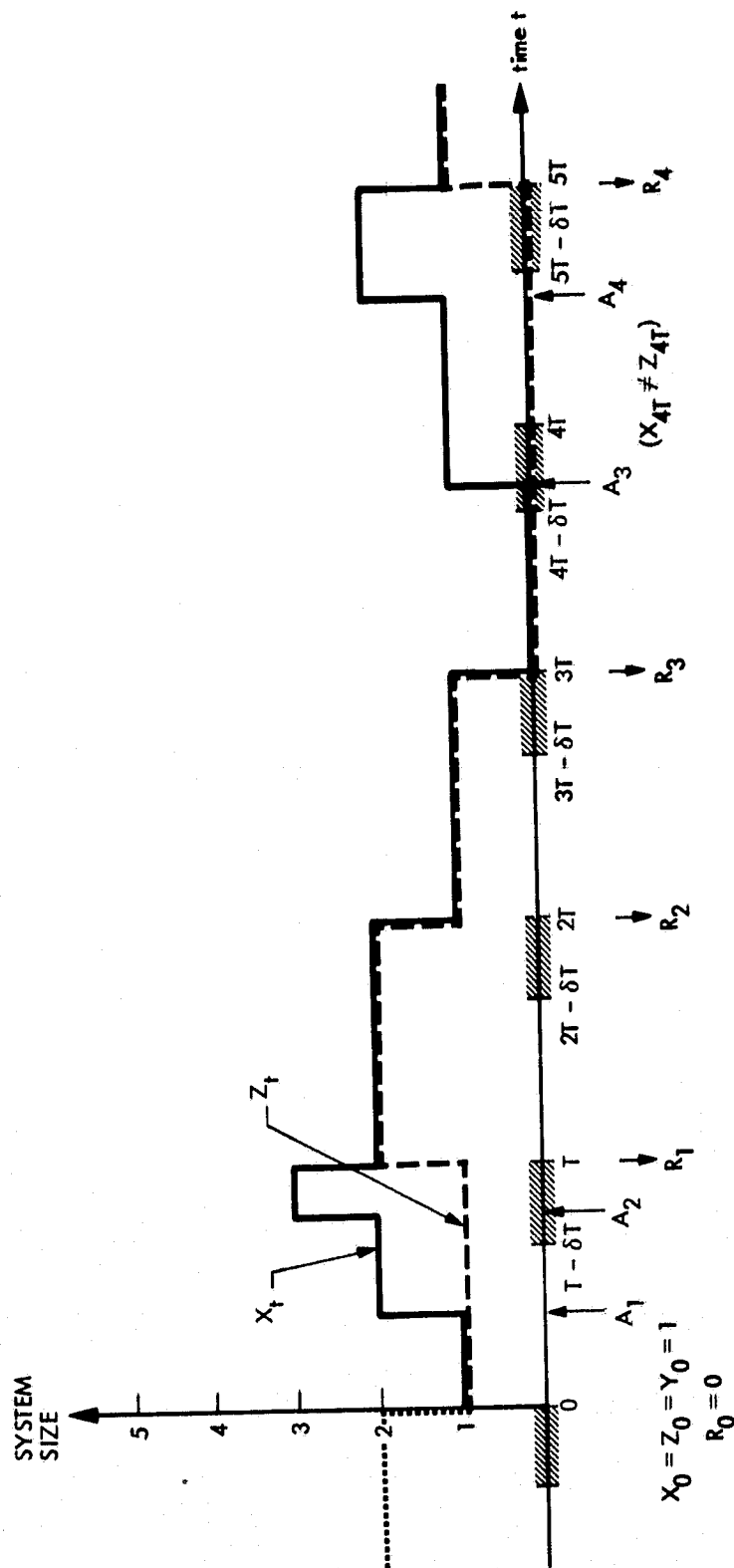


Figure 3. The Semi-Markov Process Z_t

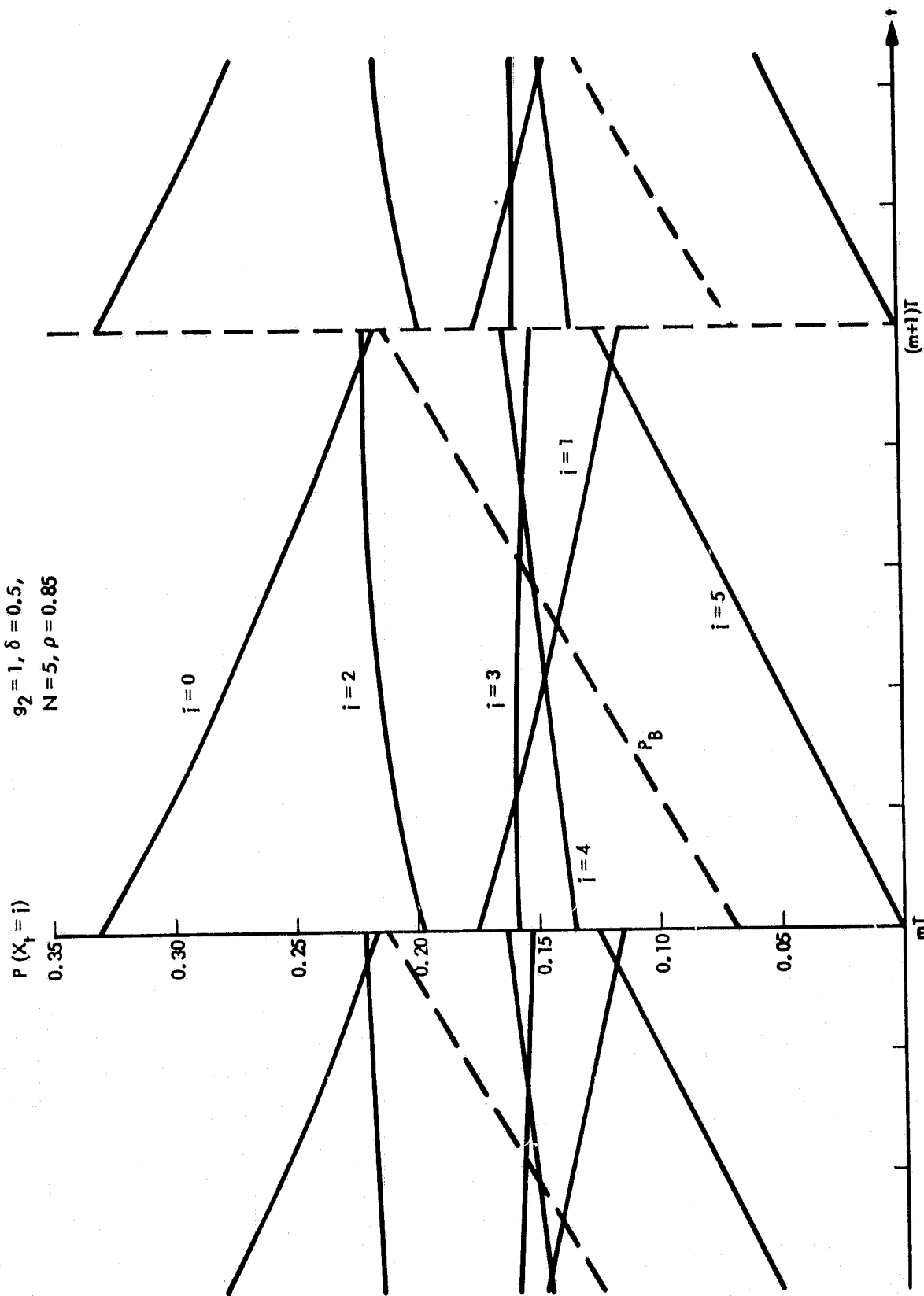


Figure 4. Packet System Size Density vs. Time

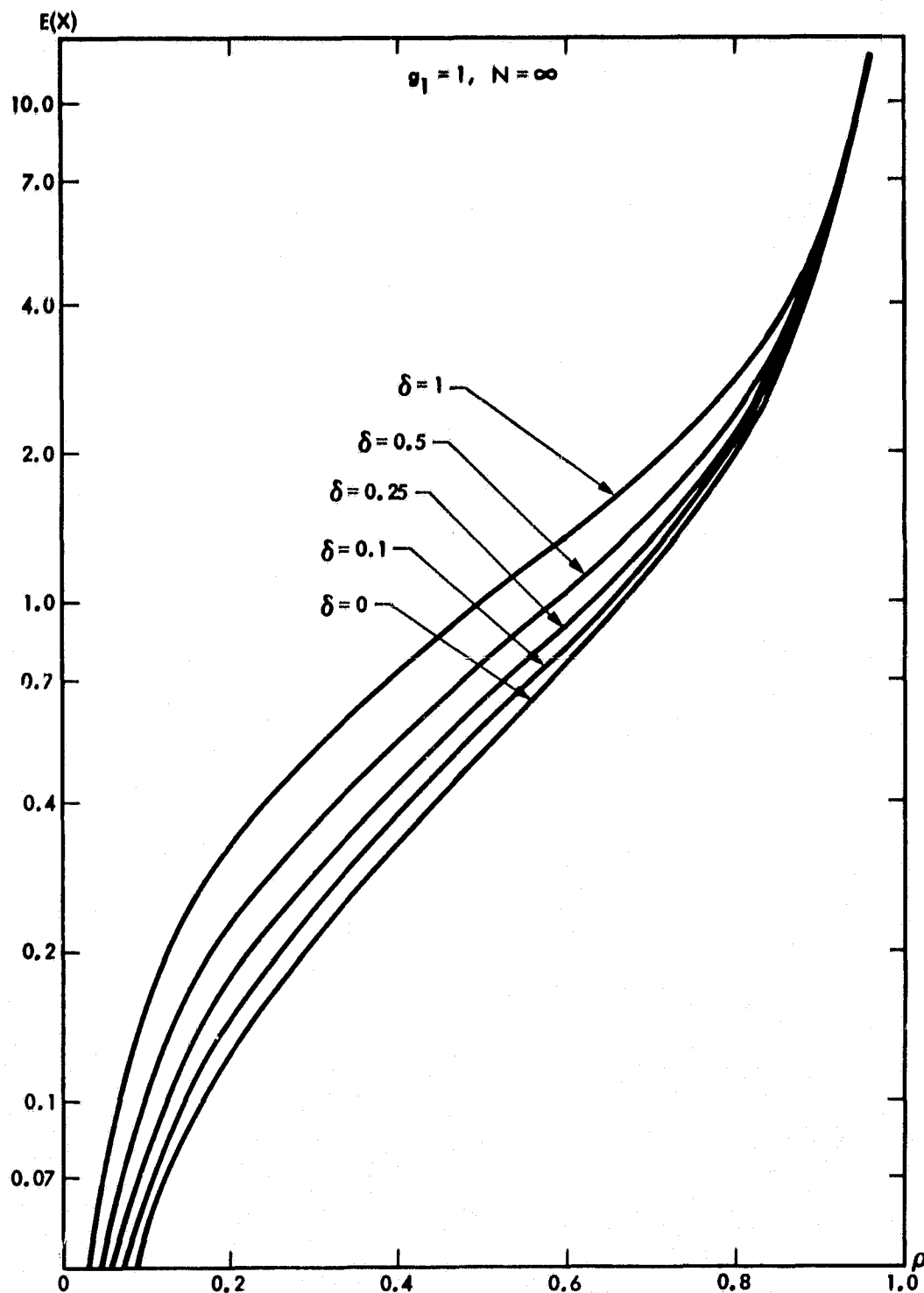


Figure 5. Mean System Size vs. Traffic Intensity for Various Normalized Slot Widths

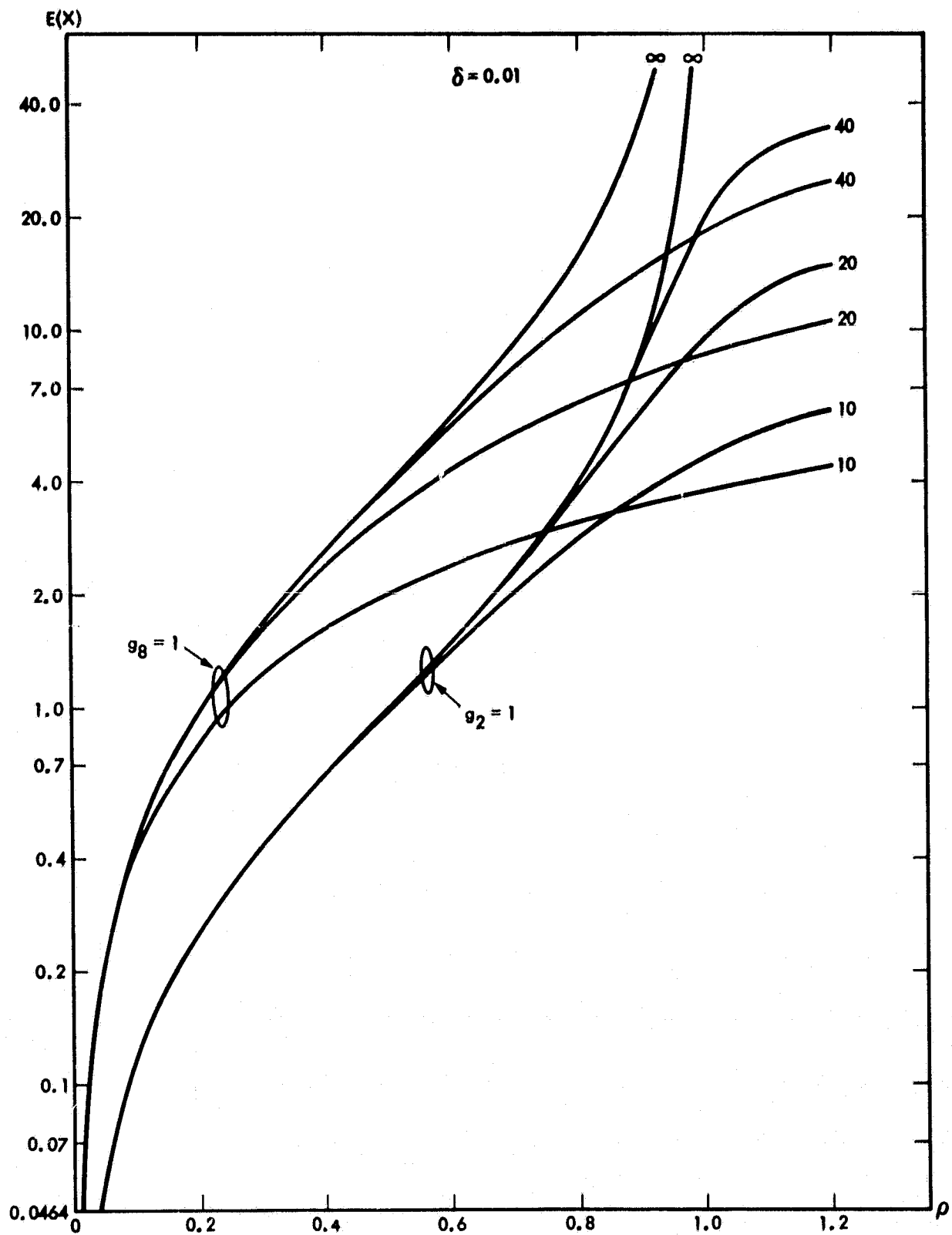


Figure 6. Mean System Size vs. Traffic Intensity for Various System Size Capacities

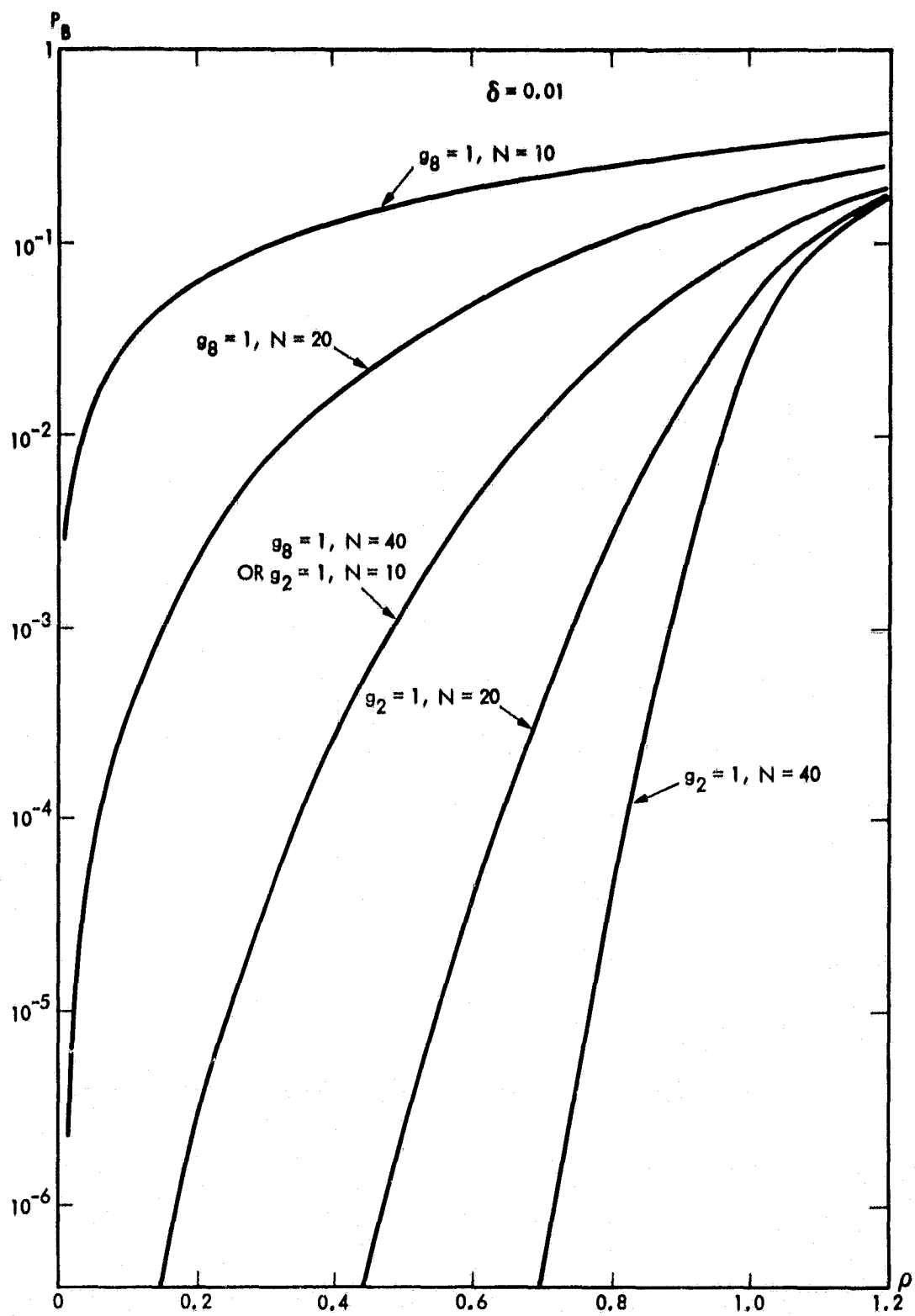


Figure 7. Packet Blocking Probability vs. Traffic Intensity

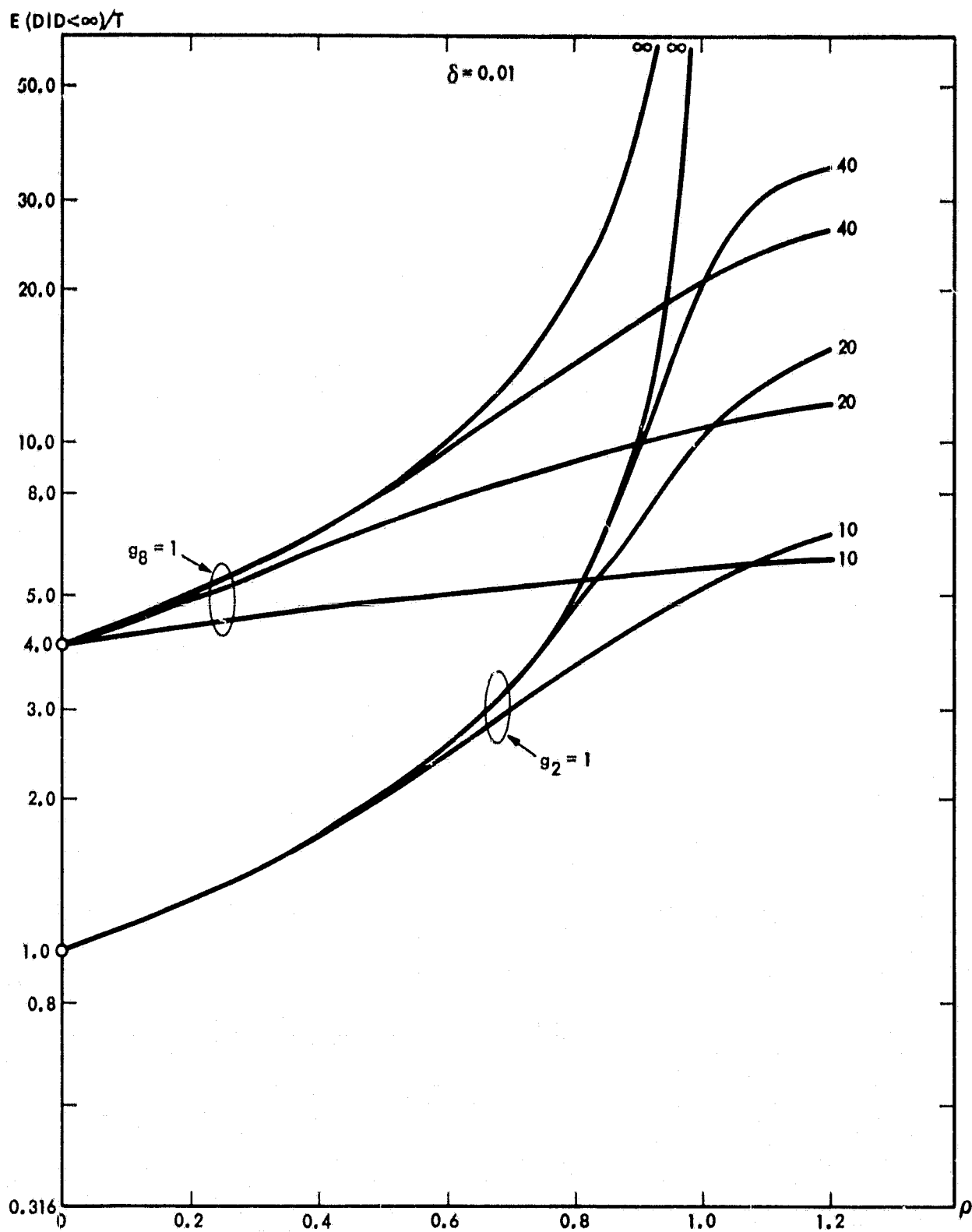


Figure 8. Mean Packet Delay vs. Traffic Intensity

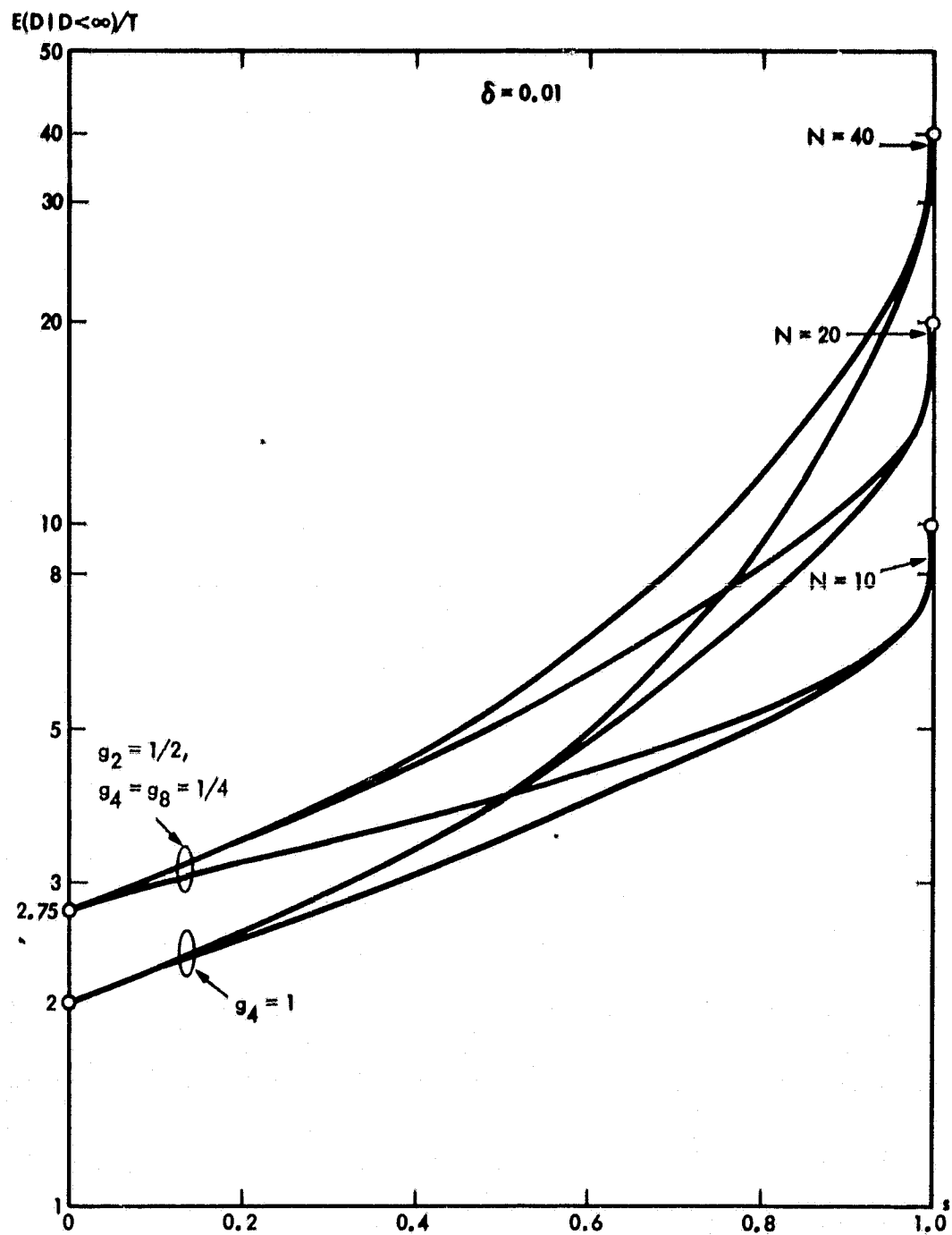


Figure 9. Mean Packet Delay (in units of frames) vs. Throughput

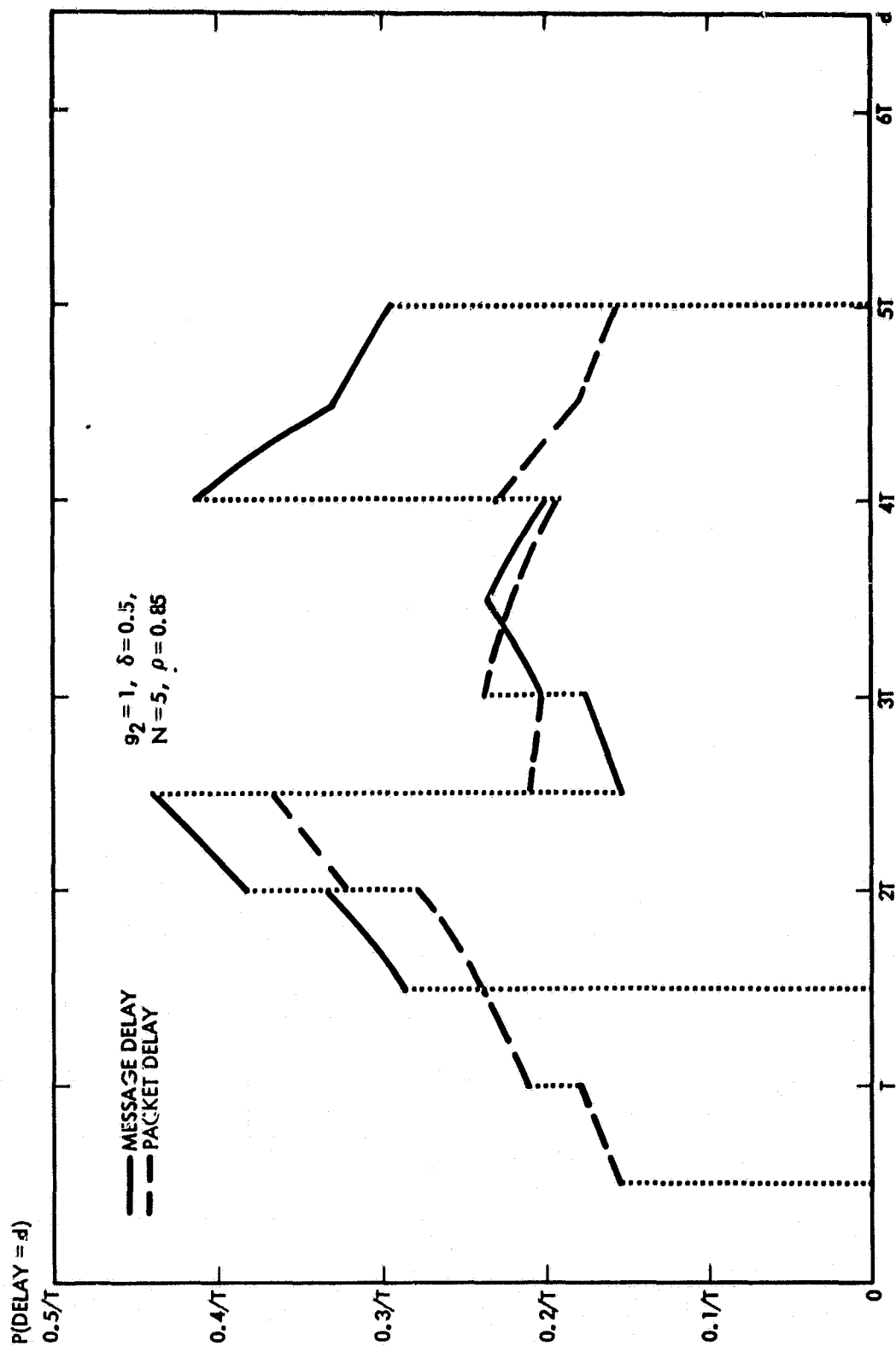


Figure 10. Actual Delay Densities